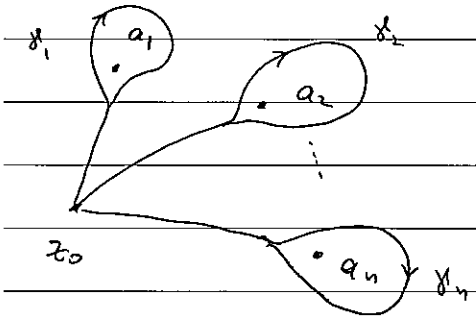
Lecture 6: The Riemann-Hilbert problem (Ch. 8 from Bolibrukh)

1. The 21-st Hilbert problem

$a_1, \dots, a_n \in \overline{\mathbb{C}} = \mathbb{P}^1$

$\chi : \pi_1(\mathbb{C} \setminus \{a_1, \dots, a_n\}) \to GL(p; \mathbb{C})$ representation

Can χ be realized as monodromy repres. of a Fuchsian system.



$G_j = \chi(\gamma_j)$, $1 \le j \le n$

Since $\gamma_n \dots \gamma_1 = 1$, we must have

$G_n G_{n-1} \dots G_1 = 1$

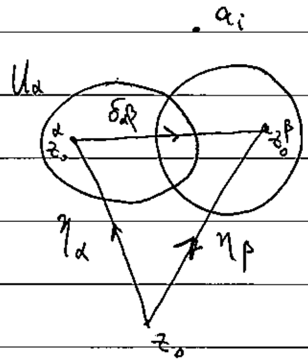
Fix χ. Try to find a merom. conn. (F, ∇)

2. Extensions

Step 1. Find a conn. (F^0, ∇) on $B = \mathbb{P}^1 \setminus \{a_1, \dots, a_n\}$ that realizes χ. Cover B by $\{U_i\}$ s.t.

(1) U_i are connected, simply connected

(2) $U_i \cap U_j$ — " —



$g_{\alpha\beta} = \chi(\eta_\alpha \circ \delta_{\alpha\beta} \circ \eta_\beta^{-1})$

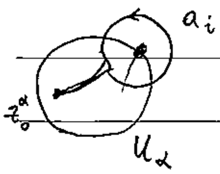
Consider $\{U_\alpha \times \mathbb{C}^p\}$ w/ $g_{\alpha\beta}$ gluing cocycles

$\nabla_\alpha = d + 0$

$\Rightarrow$ get a bundle w/ flat connection.

<u>Step 2</u>. Continue (F°, ∇) to (F, ∇) $\overset{F}{\text{holom.}}$ on $\overline{\mathbb{C}}$ s.t.
∇ merom. at $a_1, \dots, a_n$

Fix a_i, $\mathcal{U}_\alpha$ s.t. $a_i \in \overline{\mathcal{U}_\alpha}$
δ_i small loop around a_i



$(e_1^\alpha, \dots, e_p^\alpha)$ basis of horiz. sections over $\mathcal{U}_\alpha$

analytical continuation along δ_i gives
$$(e_1^\alpha, \dots, e_p^\alpha) \cdot \tilde{G}_i$$

Put $\tilde{E}_i = \frac{1}{2\pi\sqrt{-1}} \ln \tilde{G}_i$ s.t. eigenv ρ of $\tilde{E}_i$ satisfy $0 \le \operatorname{Re} \rho < 1$

Fix a branch of $(z - a_i)^{-\tilde{E}_i}$ in $\mathcal{U}_\alpha$.

$\mathcal{O}_i$ - open neighb. of a_i.
$s = (s_1, \dots, s_p)$ sections of $\mathcal{O}_i \times \mathbb{C}^p$ s.t. $[s_1 \dots s_p] = \underset{\text{identity matrix}}{I_p}$

Put $\xi^\alpha = e^\alpha \cdot (z - a_i)^{-\tilde{E}_i}$ basis of $F^\circ|_{\mathcal{U}_\alpha}$

it induces a trivializ. of $F^\circ|_{\mathring{\mathcal{O}}_i}$.

Define: $g_{\alpha 0} : \mathring{\mathcal{O}}_i \cap \mathcal{U}_\alpha \to GL_p(\mathbb{C})$ s.t. $s_i \equiv \cdot\, \xi_i^\alpha$

Note that in the trivializ. given by ξ^α the conn. takes the ~~form:~~
$$\omega = \frac{\tilde{E}_i\, dz}{z - a_i}.$$

Let $\{\tilde{e}^\alpha\}$ be another basis of horiz. sections in U_α

$(\tilde{e}^\alpha) = (e^\alpha)\cdot S$ then the monodromy becomes $\tilde{G} = S^{-1} G S$

$\Rightarrow$ we can assume the monodromy $\tilde{G}$ and $\tilde{E} = \frac{1}{2\pi\sqrt{-1}} \ln \tilde{G}$ are upper triangular.

Choose $\Lambda = \mathrm{diag}[\lambda_1, \dots, \lambda_p]$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_p \in \mathbb{Z}$ admissible matrix. Take a basis

$$\xi^{\Lambda,\tilde{\alpha}} = \tilde{e}^\alpha (z-a_i)^{-\tilde{E}_i} (z-a_i)^{-\Lambda_i}$$

single valued near $z = a_i$ $\Rightarrow$ give a trivializ. of $F^{(0)}|_{\dot{O}_i}$

$\Rightarrow$ we get an extension of $F^{(0)}$ that depends on Λ_i.

The connection matrix in the new trivializ. becomes

$$\omega^{\Lambda_i} = \left(\Lambda_i + (z-a_i)^{\Lambda_i} \tilde{E}_i (z-a_i)^{-\Lambda_i}\right) \frac{dz}{z-a_i} .$$

$\mathcal{F} = \{$ the set of all these extensions (F, ∇) for different $\Lambda_i, S_i \}$

<u>Thm.</u> $\mathcal{F}$ contains <u>all</u> extensions of (F°, ∇) to (F, ∇) so that ∇ has Fuchsian singularities at $\{a_1, \dots, a_n\}$, and monodromy repr. χ.

<u>Pf.</u> Let (F', ∇') be any bundle w/ a logarithmic connection and a monodromy repr. χ.

$$(F', \nabla')\big|_B = (F^\circ, \nabla)$$

Let (ξ) be a basis of local holom. sections of F' over $O_i \ni a_i$

$(F', \nabla')|_{O_i}$ is a Fuchsian system near $z = a_i$ $\Rightarrow$

we can choose a Levelt fundamental matrix [w/ respect to some trivializ. $\xi^\alpha = (\xi_1^\alpha, \dots, \xi_p^\alpha)$ of F' in $O_i \cap U_\alpha$]

$$Y(z) = U(z)(z-a_i)^{A_i}(z-a_i)^{E_i}$$

A_i admissible

$U(z)$ is holomorphically invertible.

The basis (s^α) of horiz. sections of ∇' over $U_\alpha \cap O_i$ w/ matrix w/ coords. $Y(z)$ (we have to fix a branch of $(z-a_i)^{E_i}$ in $U_\alpha \cap O_i$)

Define $\xi^\alpha = (\xi_1^\alpha \dots \xi_p^\alpha)$, $\xi_i^\alpha : O_i \cap U_\alpha \to \mathbb{C}^p$ by

$$(s^\alpha) = (\xi^\alpha) . Y(z) = \underbrace{(\xi^\alpha . U(z))}_{}(z-a_i)^{A_i}(z-a_i)^{\tilde{E}_i}$$

$$(\xi')^\alpha = (s^\alpha) \cdot (z-a_i)^{-\tilde{E}_i}(z-a_i)^{-A_i}$$

$\Rightarrow (F', \nabla')$ is isomorphic to a bundle from the class $\mathcal{F}$ w/ $\Lambda_i = A_i$ and $(s_1^\alpha, \dots, s_p^\alpha)$ as $\tilde{e}^\alpha$.