Preface

Vortices are often associated with dramatic circumstances such as hurricanes; this type of vortex has been studied extensively in the framework of classical fluids. Nevertheless, its quantum counterpart has gained major interest in the past few years due to the experimental realization of Bose–Einstein condensates (BEC), a new state of matter predicted by Einstein in 1925. Vortices in BEC are quantized, and their size, origin, and significance are quite different from those in normal fluids since they exemplify "superfluid" properties.

Since the first experimental achievement of Bose–Einstein condensates in 1995 in alkali gases and the award of the Nobel Prize in Physics in 2001, the properties of these gaseous quantum fluids have been the focus of international interest in condensed matter physics. This book was both motivated by this intense activity, especially in the group of Jean Dalibard at the Ecole normale supérieure, but also by the constant development of mathematical techniques which could prove useful in tackling these problems, in particular in the group of Haim Brezis. This monograph is dedicated to the mathematical modelling of some specific experiments which display vortices and to a rigorous analysis of features emerging experimentally. It can serve as a reference for mathematical researchers and theoretical physicists interested in superfluidity and quantum condensates, and can also complement a graduate seminar in elliptic PDEs or modelling of physical experiments. There are two introductory chapters: the first is related to the physics background, while the second is devoted to the presentation of the mathematical results described in the book.

Vortices have been observed experimentally by rotating the trap holding the atoms in the condensate. In contrast to a classical fluid, for which the equilibrium velocity corresponds to solid body rotation, a quantum fluid such as a Bose–Einstein condensate can rotate only through the nucleation of quantized vortices beyond some critical velocity. There are two interesting regimes: one close to the critical velocity where there is only one vortex, and another at high rotation values, for which a dense lattice is observed. Another experiment consists of a superfluid flow around an obstacle: at low velocity, the flow is stationary; while at larger velocity, vortices are nucleated from the boundary of the obstacle.

One of the key issues is thus the existence of these quantized vortices. We address this issue mathematically and derive information on their shape, number, and location. In the dilute limit of the experiments, the condensate is well described by a mean field theory and a macroscopic wave function, solving the so-called Gross–Pitaevskii equation. The mathematical tools employed are energy estimates, Gamma convergence, and homogenization techniques. We prove existence of solutions which have properties consistent with the experimental observations. Open problems related to recent experiments are also presented. They will require the development of new tools related for instance, to microlocal analysis or time–dependent problems.

The suggestion for setting down these important ideas came from Haim Brezis, and I would like to thank him warmly for his constant enthusiasm and support while I was working on it. Many tools used here have been developed by either him or his school. I am glad to be able to present an application of this beautiful mathematics to today's physics.

I am also extremely grateful to Jean Dalibard, who has always been willing to take time to share his experiments, his ideas, and his interests in how mathematics can contribute to physics. Working together with him and writing a joint paper was a real pleasure and a source of mathematical problems for many years to come. I would also like to thank him for his careful reading of this manuscript. Before working with Jean, I had the opportunity of many fruitful discussions with members of his group, in particular Vincent Bretin, Yvan Castin, and David Guéry-Odelin. I have always appreciated their open minds and interest in mathematics. I would like to thank David in particular for his comments leading to improvements in the introductory parts of this book.

I owe my personal interest in interdisciplinary topics to the joint efforts of Etienne Guyon, Yves Pomeau, and Henri Berestycki, who launched a program for students at the Ecole normale supérieure to spark interest in problems on the border between mathematics and physics. This effort was a real success, as were the various maths-physics meetings in Foljuif, a property of the Ecole normale supérieure. At one of them, I met Yvan Castin and realized that we had mathematical tools that could help in understanding problems emerging in rotating Bose–Einstein condensates. I would like to again my deep gratitude to Etienne Guyon, Yves Pomeau, and my supervisor Henri Berestycki, for all that I discovered has been thanks to them.

I would like to also thank, of course, all my collaborators on these topics, in particular: Tristan Rivière, with whom this huge program started and the evidence of vortex bending occurred; Bob Jerrard, whose involvement in understanding the shape of vortices was quite influential and with whom it was a pleasure to work in Vancouver, Milan, Istanbul, and Minneapolis (I thank all the hosting institutions) and who has undertaken a very careful reading of the book; Qiang Du and Ionut Danaila, who have performed, on different topics, beautiful numerical computations; Stan Alama and Lia Bronsard, who came to Paris and became interested in these topics when they discovered them; Xavier Blanc, with whom I am very happy to have worked with on many projects; and very recently Francis Nier, who has allowed me to discover microlocal analysis and Bargmann transforms, which have proved to

be quite useful in tackling these problems. It was also very rewarding to work with outstanding physicists, Jean Dalibard and Yves Pomeau. Many colleagues all over the world have mentioned that I was quite lucky to have had this opportunity and to have found a common language to speak. I certainly believe it.

Part of this monograph was taught as a Ph.D. course at Paris 6 in 2003–2004. One of the results presented here was obtained by two of these Ph.D. students, Radu Ignat and Vincent Millot, whom I jointly supervised with Haim Brezis. I am pleased to describe their work in one of the chapters.

The quality of the presentation of this book was greatly improved thanks to all the lectures given in various universities or summer schools. I would like to thank in particular: Luis Caffarelli and Irene Gamba, Peter Constantin, Peter Sternberg, Miguel Escobedo, Gero Friesecke, Fang-Hua Lin, Stefan Muller, Tristan Rivière, and Juan-Luis Vazquez. I have also benefited from informal discussions with Fabrice Bethuel, Petru Mironescu, Sylvia Serfaty, Etienne Sandier, and Didier Smets.

I take the opportunity here to express my gratitude to all my colleagues in the Laboratoire Jacques-Louis Lions, in particular: to Yvon Maday, who is a very enthusiastic head of department; to my office mate Xavier Blanc; to my office neighbours, Edwige Godlewski and Francois Murat; to Olivier Glass, Frédéric Hecht, Simon Masnou, and the staff in the laboratory, Danielle Boulic, Michel Legendre, Jacques Portes, and Liliane Ruprecht.

The writing of this book was made possible by my position at CNRS, which should be naturally associated with the outcome of such interdisciplinary effort. My research during this period was supported by a CNRS grant for young researchers and a French ministry of research grant, ACI "Nouvelles interfaces des mathématiques." Some of the open problems were derived during a maths–physics meeting organized with David Guéry-Odelin at the "Fondation des Treilles" in Tourtour, and I would like to acknowledge their welcome.

Finally, I would like to thank my family and friends for their constant support in the preparation of the manuscript and all the people at Birkhäuser, in particular Ann Kostant, for their help.

Paris, October 31, 2005, *Amandine Aftalion*

Contents

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The Physical Experiments and Their Mathematical Modelling

Bose–Einstein condensation (BEC), first predicted by Einstein in 1925, has been realized experimentally in 1995 in alkali gases. The award of the 2001 Nobel Prize in Physics to E. Cornell, C. Wieman, and W. Ketterle acknowledged the importance of the achievement. In this new state of matter, which is very dilute and at very low temperature, a macroscopic fraction of the atoms occupy the same quantum level, and behave as a coherent matter wave similar to the coherent light wave produced by a laser. In the dilute limit, the condensate is well described by a mean-field theory and a macroscopic wave function. The properties of these gaseous quantum fluids have been the focus of international interest in physics, both experimentally and theoretically, and many applications are envisioned. An important issue is the relationship between BEC and superfluidity, in particular through the existence of vortices. The focus of this book is the mathematical properties of vortices, observed in very recent experiments on rotating condensates.

After a brief summary of the main achievements leading to BEC, we will describe the experimental device which has had an influence on the mathematical modelling through the trapping potential. Then we will focus on a specific experiment on rotating condensates and the main observations that have been made. Finally, we will state the mathematical framework and explain the issues that we want to address in this book.

1.1 A hint on the experiments

1.1.1 Brief historical summary

The phenomenon of condensation was predicted in 1925 by Einstein, on the basis of a paper by Bose: for a gas of noninteracting particles, below a certain temperature there is a phase transition where a macroscopic fraction of the gas gets condensed, that is, a significant fraction of the atoms occupy the state of lowest energy. This quantum degeneracy, which is a consequence of statistical effects of atoms in a box, occurs when the interatomic distance becomes comparable to the de Broglie wavelength, given by $\lambda_{\mathrm{dB}} = h/(2\pi m k_{\mathrm{B}} T)^{1/2}$, where T is the temperature, k_{B}, the Boltzmann constant, and m the mass of each atom. This condition implies that the gas is at very low temperature. In 1925, there was no experimental evidence of the phenomenon, since at the temperature required by the theory, all known materials were in the solid state.

In 1938, after the discovery of the superfluidity of liquid helium independently by Allen and Misener [20], and Kapitza [90], London [103] made a link between superfluidity and Einstein's theory. But in liquid helium, less than 10% of the atoms are condensed and the system is strongly interacting, while Einstein's model is for an ideal gas.

Intense theoretical work was done, in particular by Bogoliubov, to understand the relationship between superfluidity and Bose–Einstein condensation. The prediction of quantized vortices was made by Onsager [118] in 1949 and Feynman [63] in 1955, with the experimental discovery by Hall and Vinen [77] in 1956 and the direct observation by Packard and Sanders [119] in 1972.

The fact that interactions in helium reduce the occupancy of the lowest energy state led to the search for substances closer to Einstein's ideal gas model, which can be produced in a metastable dilute phase and lead to a high condensate fraction.

The essential techniques to produce quantum degenerate gases are cooling techniques, and in particular laser cooling. The 1997 Nobel Prize in physics was awarded for the development of these techniques [48]. Their improvement has led, in 1995, to the achievement of Bose–Einstein condensation in atomic gases by the Jila group in Boulder (Cornell, Wiemann [51]), and very soon afterwards by the MIT group (Ketterle [92]). A BEC is a quantum macroscopic object which can be described by a complex-valued wave function. There are many different experiments being made on this new state of matter; we refer to the books by Pethick–Smith [120] and Pitaevskii–Stringari [124] for more details on the stakes and the theoretical approach. We point out that although the gases are dilute, interactions play an important role.

The main concern of this book lies in experiments on rotating condensates, which are similar to experiments on helium and allow the observation of quantized vortices.

1.1.2 How to achieve a BEC experimentally

There are several steps in the achievement of condensation. The first is laser cooling, achieved with three pairs of counterpropagating laser beams along three orthogonal axes. The gas is precooled, so that it can be confined in a magnetic trap (Ioffe–Pritchard trap) as illustrated in Figure 1.1. The temperature reached is of order 100 μK, with 10^9 atoms in a volume on the order of one cm^3. Laser cooling alone cannot produce sufficiently high density and low temperature for condensation. The second step, evaporative cooling (in some sense similar to blowing on your coffee to cool it), allows one to remove the most energetic atoms and thus further cool down the cloud. The drawback is that a large number of atoms are lost in the process: at the end, there are about 10^4–10^7 atoms, and the final temperature is below 1μK. For a more detailed description, one may refer to the Nobel lectures [48, 51, 92] or to [25, 53, 120].

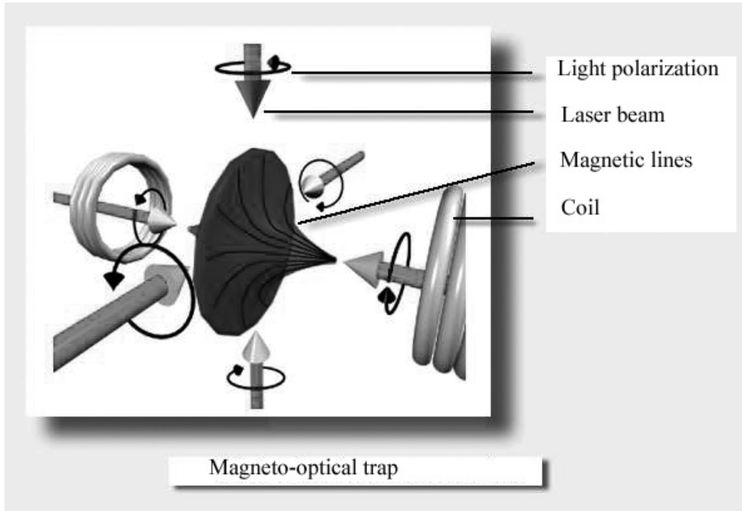


Fig. 1.1. Experimental realization of a trap. Courtesy of V. Bretin.

The important feature for our mathematical purposes is the confinement of the atoms: a term representing the magnetic trapping potential $m(\omega_x^2 x^2+\omega_y^2 y^2+\omega_z^2 z^2)/2$ (where ω_i are the trap frequencies in the i direction) will appear in the energy and the equations. This trapping potential is also at the origin of the nonuniformity of the gas and the special shape of the condensate: cigar-shaped, as illustrated in Figure 1.2.

1.1.3 Experimental observations

The experiment that we want to focus on is reminiscent of the classical rotating bucket experiment for superfluid helium [57, 119]. When a normal fluid is rotated in a bucket at velocity $\tilde{\mathbf{\Omega}}$, the fluid rotates as a rigid body, that is, the velocity $\mathbf{v}$ increases smoothly from the center to the edges, $\mathbf{v} = \tilde{\mathbf{\Omega}} \times \mathbf{r}$, and the flow is characterized by

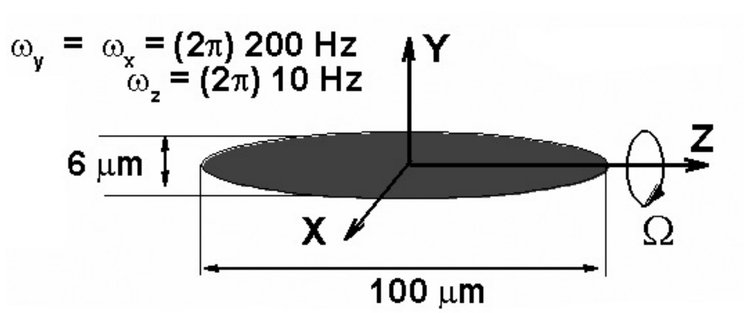


Fig. 1.2. Cigar-shaped condensate. Courtesy of V. Bretin.

uniform vorticity $\nabla \times \mathbf{v} = 2\tilde{\mathbf{\Omega}}$. In a quantum fluid, the velocity field is irrotational almost everywhere. This is a consequence of the description by a complex-valued order parameter $\psi = |\psi|e^{iS}$: ∇S is identified to a velocity $\mathbf{v}$, and thus $\nabla \times \mathbf{v} = 0$ as soon as ψ does not vanish. The zeroes of ψ around which there is a circulation of phase are the singularities or vortices. In order to rotate, the flow field has to develop singularities.

Quantized vortices have been obtained using different approaches. A first device consists in imprinting the phase of the condensate to drive its rotation by optical means [111]; it was used to investigate the precession of the vortex core around the symmetry axis of the trap. The technique on which we will focus resembles the rotating bucket experiment: a rotating laser beam superimposed on the magnetic trap allows one to spin up the condensate by creating a harmonic anisotropic rotating potential along the z axis (see Figure 1.2). This has been developed in the groups of J. Dalibard at the ENS in Paris [107, 108, 132] and W. Ketterle at MIT [1, 127].

For small angular velocities, there is no modification of the condensate. For sufficiently large angular velocities, vortices appear in the system. These vortices correspond to permanent currents, whose existence is a consequence of the superfluidity. In Figure 1.3, black regions correspond to the singularity lines or vortices: this is a view of the cross section of the condensate (xy plane) integrated on z. The white region corresponds to places where $|\psi|$ is significant. For low velocity, there is a small number of vortices, while at larger velocity a triangular lattice is observed, called the Abrikosov lattice, since it is reminiscent of the physics of superconductors. Vortices are imaged after expansion of the condensate: in Figure 1.4, the condensate undergoes a free fall. This time-of-flight technique acts as a microscope, magnifying the size of the vortex as well as that of the condensate by a factor up to 30. This modifies the length in the x and y directions.

We do not discuss the dynamical mechanism of nucleating vortices. Depending on whether $\tilde{\mathbf{\Omega}}$ is turned on rapidly or increased adiabatically, the instabilities do not occur in the same way and hysteresis phenomena may occur. A condensate with multiple vortices is created by rotating at a resonant frequency. During the next two seconds, the rotation is stopped and the vortex lattice decays. Eventually, the condensate is left with a single vortex, whose lifetime is on the order of a few seconds. We are interested in stable configurations according to the value of the rotation.

A special geometrical feature of the singularities is their three-dimensional shape: the lines are not straight along the axis of rotation but bent as illustrated in

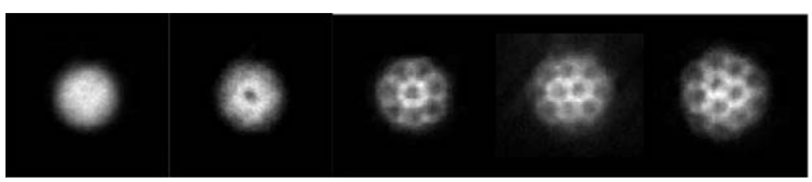


Fig. 1.3. Transverse imaging in the xy plane: from left to right, the rotational velocity increases. Courtesy of V. Bretin and J. Dalibard.

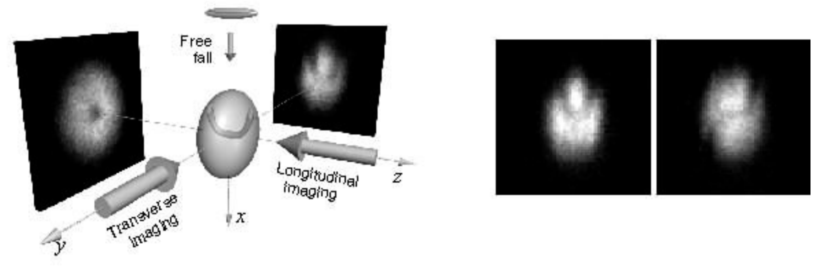


Fig. 1.4. Transverse and longitudinal imaging of a vortex (left). Single vortex lines (right). Courtesy of V. Bretin and J. Dalibard.

Figure 1.4 (experimental result). The longitudinal imaging (along the z axis) reproduces what is observed in Figure 1.3. The transverse imaging allows us to see that the vortex line is not straight. The shape is distorted with respect to the initial one, because of the imaging technique.

1.2 The mathematical framework

The aim of this book is to present a mathematical framework and theoretical elements in order to justify these observations rigorously. In particular, we would like to address the issues of the shape of the first vortex line, the critical rotational velocities for the nucleation of vortices, and the existence and properties of the lattice.

1.2.1 The Gross–Pitaevskii energy

The study of interacting nonuniform dilute gases at zero temperature can be made in the framework of the Gross–Pitaevskii energy. This means that the field operator, used to describe quantum phenomena can be replaced by a classical field $\psi(r, t)$, also called the order parameter or wave function of the condensate. This relies on the following assumptions:

- A large fraction of atoms are in the same state.
- The potential describing the interaction between atoms can be replaced by a model potential, reproducing the same scattering properties, which can be handled in the Born approximation. The scattering length a is an important datum, since it can be measured experimentally.
- The wave function varies slowly on distances of order the range of interatomic forces.

These assumptions are indeed fulfilled for condensed gases and this model is very satisfactory for describing the experiments mentioned above. We refer to the books

[120] or [124] for more details. The rigorous derivation of the Gross–Pitaevskii energy from the many-body Hamiltonian was made, in some asymptotic limit, by Lieb, Seiringer, and Yngvason [99] in the nonrotating case and very recently by Lieb and Seiringer [98] in the rotating case. This type of issue will not be addressed here; throughout the book, the mean-field description by the Gross–Pitaevskii energy will be used and the properties of its minimizers will be analyzed.

We are interested in stationary phenomena: thus in the frame rotating at angular velocity $\tilde{\mathbf{\Omega}} = \tilde{\Omega}\mathbf{e}_z$, the trapping potential and the wave function are time-independent. The wave function ψ minimizes the following energy, called the Gross–Pitaevskii energy, which includes in this order a kinetic contribution, a term due to rotation, a term due to the presence of the harmonic trapping (ω_i is the trapping frequency in the i direction), and a term due to the atomic interaction (we denote by N the number of atoms in the system and $g_{3D} = 4\pi\hbar^2 a/m$, where a is the scattering length mentioned above):

$$\mathcal{E}_{3D}(\psi) = \int_{\mathbf{R}^3} \frac{\hbar^2}{2m}|\nabla\psi|^2 \\ -\frac{\hbar}{2}\tilde{\mathbf{\Omega}} \times \mathbf{x} \cdot (i\psi\nabla\bar{\psi} - i\bar{\psi}\nabla\psi) + \frac{m}{2}V(\mathbf{x})|\psi|^2 + \frac{N}{2}g_{3D}|\psi|^4, \tag{1.1}$$

with $V(\mathbf{x}) = \omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2$, under the constraint $\int |\psi|^2 = 1$. Here, m is the atomic mass, $\mathbf{x} = (x, y, z)$, $\tilde{\mathbf{\Omega}} \times \mathbf{x}$ is the wedge product of the two vectors and is thus equal to $(-\tilde{\Omega}y, \tilde{\Omega}x, 0)$. In what follows, we will denote the vector $(i\psi\nabla\bar{\psi} - i\bar{\psi}\nabla\psi)/2$ by $(i\psi, \nabla\psi)$, which has the meaning of a scalar product in $\mathbf{C}$.

We call $d = (\hbar/m\omega_x)^{1/2}$ the characteristic length of the harmonic oscillator and assume $\omega_y = \alpha\omega_x$, $\omega_z = \beta\omega_x$. Rescaling distances by d, that is, setting $\phi(\mathbf{x}) = d^{3/2}\psi(d\mathbf{x})$, the energy becomes, in units of $\hbar\omega_x$,

$$\int_{\mathbf{R}^3} \frac{1}{2}|\nabla\phi|^2 - \frac{\tilde{\mathbf{\Omega}}}{\omega_x} \times \mathbf{x} \cdot (i\phi, \nabla\phi) + \frac{1}{2}(x^2 + \alpha^2 y^2 + \beta^2 z^2)|\phi|^2 + 2\pi N \frac{a}{d}|\phi|^4 \tag{1.2}$$

under $\int |\phi|^2 = 1$. If $\tilde{\Omega} > \min(\omega_x, \omega_y)$, the energy is not bounded below: the rotating force is stronger than the trapping potential.

There are three distinct regimes of rotation according to the value of $\tilde{\Omega}$. For low rotational velocity, there are no vortices in the system. Then for intermediate $\tilde{\Omega}$, there are a few vortices arranged in the xy plane as illustrated in Figure 1.3. Their characteristic size is much smaller than their interdistance. Thus, we will introduce a small parameter describing what is called the Thomas–Fermi regime. We will make an expansion of the energy in terms of this parameter, which will allow us to identify the critical value of the velocity as well as the location and shape of vortices. We will see that, in this regime, the condensate and the wave function are localized in the domain $\mathcal{D}$ defined by

$$\mathcal{D} = \{x^2 + \alpha^2 y^2 + \beta^2 z^2 < \rho_0\}, \tag{1.3}$$

where ρ_0 is determined by

$$\int_{\mathcal{D}} \rho_{\mathrm{TF}}(\mathbf{r}) = 1 \text{ where } \rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - (x^2 + \alpha^2 y^2 + \beta^2 z^2). \tag{1.4}$$

This yields $\rho_0^{5/2} = 15\alpha\beta/8\pi$.

The other regime is at high rotational velocity, that is, when $\tilde{\Omega}$ is close to the trapping frequency ω_x (we call $\Omega = \tilde{\Omega}/\omega_x$). Then, the centrifugal force nearly balances the trapping force, the condensate expands, and the number of vortices diverges. There is a dense lattice for which vortices have approximately the same size as their interdistance. The condensate is no longer localized in $\mathcal{D}$ but in a larger domain $\mathcal{D}_1 = \{x^2(1-\Omega^2) + (\alpha^2 - \Omega^2)y^2 + \beta^2 z^2 < \rho_1\}$, where ρ_1 is defined by some normalization condition, similar to ρ_0. This is due to the fact that the centrifugal force creates an effective trapping potential, whose frequencies in the x and y directions are $\omega_x\sqrt{1-\Omega^2}$ and $\omega_x\sqrt{\alpha^2-\Omega^2}$. The study of the lattice will rely on homogenization techniques and double-scale convergence.

Though the shape of vortices is interesting from a three-dimensional point of view, before studying the full three-dimensional model related to the experiments, we want to restrict to a simpler situation in two dimensions, which allows us to understand the main features. Our two-dimensional reduction corresponds either to a condensate which is an infinite cylinder (then the coefficient a in (1.2) has to be replaced by a/Z, where Z is the elongation of the condensate), or to a condensate very thin in the z direction (that is, when β is large). In this latter case, the two-dimensional model can be rigorously derived from the three-dimensional energy. We refer to Schnee, Yngvason [140] and Olshanii [117] for details: when the confinement in the z direction is much stronger, the wave function $\phi(x, y, z)$ can be decoupled into $\phi_0(x, y)\xi(z)$; $\xi(z)$ corresponds to the ground state without interaction in the z direction and is a Gaussian, and ϕ_0 minimizes the corresponding 2D problem, where the scattering length a has been modified to include the reduction, that is, $a_{2d} = a/a_z$, where $a_z = (\hbar/m\omega_z)^{1/2}$. In this setting, the function $\rho_{\mathrm{TF}}(x, y)$ is equal to $\rho_0 - x^2 - \alpha^2 y^2$, and the domain $\mathcal{D} = \{\rho_{\mathrm{TF}} > 0\}$ is two-dimensional. This two-dimensional reduction is particularly meaningful in the case of high rotational velocity, since the ratio of the trapping frequency along the z direction and the effective perpendicular trapping frequency are of order $\sqrt{1-\Omega}$. This ratio becomes small when Ω reaches the trapping frequency ω_x. We will use this reduction in this setting, though it is an open problem to derive it rigorously.

1.2.2 The Thomas–Fermi regime

A limit often considered in the literature, and called the Thomas–Fermi regime, occurs when the kinetic energy is small in front of the trapping and interaction terms. This is the limit that we are going to consider for the case of low velocity, when there is a small number of vortices in the system. This limit can be quantified by the introduction of a small parameter

$$\varepsilon = \left(\frac{d}{8\pi N a} \right)^{2/5}. \tag{1.5}$$

Using the experimental values of the ENS group [108] for rubidium atoms,[1] we find that $\varepsilon = 2.75\ 10^{-3}$. We rescale the distances by $R = d/\sqrt{\varepsilon}$ and define $u(\mathbf{r}) = R^{3/2}\phi(\mathbf{x})$, where $\mathbf{x} = R\mathbf{r}$, and we set $\Omega = \tilde{\Omega}/\varepsilon\omega_x$. The energy can be rewritten as

$$\int_{\mathbf{R}^3} \frac{1}{2}|\nabla u|^2 - \boldsymbol{\Omega} \times \mathbf{r} \cdot (iu, \nabla u) + \frac{1}{2\varepsilon^2}(x^2 + \alpha^2 y^2 + \beta^2 z^2)|u|^2 + \frac{1}{4\varepsilon^2}|u|^4. \tag{1.6}$$

Due to the constraint $\int |u|^2 = 1$, we can add any multiple of $\int |u|^2$, so that it is equivalent to minimize

$$E_\varepsilon(u) = \int_{\mathbf{R}^3} \frac{1}{2}|\nabla u|^2 - \boldsymbol{\Omega} \times \mathbf{r} \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2}|u|^4 - \frac{1}{2\varepsilon^2}\rho_{\mathrm{TF}}(\mathbf{r})|u|^2, \tag{1.7}$$

where ρ_{TF} is defined by (1.4). If Ω is bounded above by $C|\log \varepsilon|$, one can check that if $\mathbf{r} \in \mathbf{R}^3 \setminus \mathcal{D}$, since the energy is convex in this region, $|u|$ decays exponentially fast [80]. This means that $\mathcal{D}$ corresponds to the region where $|u|$ is significant, that is, the location of the condensate in the figures. The domain $\mathcal{D}$ is elongated in the z direction because β is much smaller than 1 in the experiments.

In order to simplify the mathematics, we will restrict to minimizing E_ε in $\mathcal{D}$ instead of $\mathbf{R}^3$, with the condition $u \in H_0^1(\mathcal{D})$, and we will ignore the L^2 constraint, which will be almost satisfied because of (1.4). More details will be given in the last paragraph of this section.

Thus, we will be interested, according to the value of Ω, both when $\mathcal{D}$ is an ellipsoid in $\mathbf{R}^3$ and when it is an ellipse in $\mathbf{R}^2$, in the minimizers $u \in H_0^1(\mathcal{D})$ of

$$E_\varepsilon(u) = \int_{\mathcal{D}} \frac{1}{2}|\nabla u|^2 - \boldsymbol{\Omega} \times \mathbf{r} \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2}(|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}))^2 \tag{1.8}$$

where $\mathcal{D}$ and ρ_{TF} are respectively given by (1.3) and (1.4), $\boldsymbol{\Omega} = \Omega e_z$, and $\mathbf{r} = (x, y, z)$. We point out that another way to write the energy is

$$E_\varepsilon(u) = \int_{\mathcal{D}} \frac{1}{2}|\nabla u - i\mathbf{A}u|^2 - \frac{1}{2}\boldsymbol{\Omega}^2 r^2|u|^2 + \frac{1}{4\varepsilon^2}(|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}))^2, \tag{1.9}$$

where $\mathbf{A} = \boldsymbol{\Omega} \times \mathbf{r}$. This formulation is reminiscent of the one used for the Ginzburg–Landau model of superconductors [109, 138], and so is the value of the critical rotational velocity. Yet the nonuniformity of ρ_{TF} gives rise to a new series of phenomena and special patterns for vortices.

The mathematical techniques are inspired by the tools developed for the analysis of vortices in the framework of the Ginzburg–Landau problems by Bethuel–Brezis–Helein [32] in 2D and Riviere [131] in 3D, Jerrard [84], Jerrard–Soner [86], and

[1] $m = 1.445 \times 10^{-25}$kg, $a = 5.3 \times 10^{-9}$m, $N = 1.4 \times 10^5$, and $\omega_x = 1094\,\mathrm{s}^{-1}$ with $\alpha = 1.06$, $\beta = 0.067$.

Sandier–Serfaty [138]. We will see that the critical velocity of nucleation of the first vortex is of order $|\log \varepsilon|$. We will prove that $|u|^2$ is very close to ρ_{TF} except in tubes in 3D or balls in 2D of characteristic size ε where u vanishes. Close to the first critical velocity, the number of vortices is bounded. The distances between vortices, of order $1/\sqrt{|\log \varepsilon|}$, is much larger than their characteristic size ε. The location and number of these singularities will be identified through an energy expansion.

We are now going to describe briefly the most significant results of this book concerning the Thomas–Fermi regime. More rigorous statements will be made in the next chapter, which is a mathematical introduction. One of the main results is the derivation of a reduced line energy, which allows us to justify the experimental observation of a bent vortex in Figure 1.4 (see also our numerical computations in Figure 1.5, which will be described in more detail in Chapter 6). If the problem is reduced to a two–dimensional setting, the expansion of the energy in terms of ε can be made more precise and we are able to determine the critical velocity for the existence of n vortices. Finally, we present results for other types of trapping potentials.

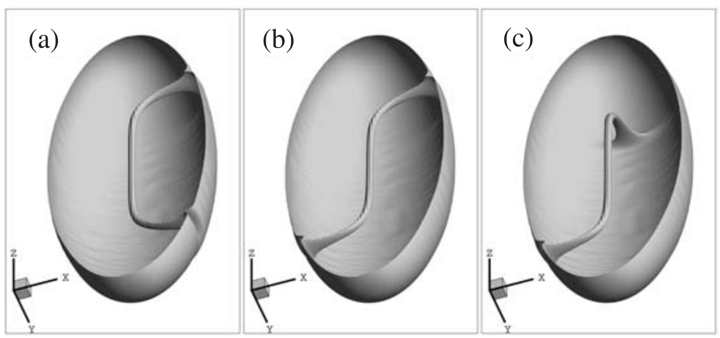


Fig. 1.5. Single-vortex configurations in BEC: (a) U vortex, (b) planar S vortex, (c) nonplanar S vortex. Isosurfaces of lowest density within the condensate.

The bent vortex

This is one of the main results of this book, which will be described in Chapter 6. When the domain $\mathcal{D}$ defined by (1.3) is three-dimensional, the shape of vortices depends on the elongation β in the z direction. We are able to determine the critical velocity for which vortices are favorable, and their optimal shape. Vortices can be represented by oriented curves, that is, Lipschitz functions $\gamma\colon (0, 1) \to \mathcal{D}$. They correspond to the zero set of u around which there is a circulation. We prove that if the minimizer has a singularity line γ, then the energy E_ε has the following expansion:

$$E_\varepsilon(u) = \mathcal{E}_0(\varepsilon) + \pi|\log \varepsilon|\mathcal{E}_\gamma + o(|\log \varepsilon|),$$

where $\mathcal{E}_0(\varepsilon)$ is the the leading-order term and the energy of the vortex-free solution, which does not depend on the solution u, while the energy of the vortex line $\mathcal{E}_\gamma$ is

$$\mathcal{E}_\gamma = \int_\gamma \rho_{\mathrm{TF}}\, dl - \frac{\Omega}{(1+\alpha^2)|\log\varepsilon|}\int_\gamma \rho_{\mathrm{TF}}^2\, dz. \tag{1.10}$$

The energy $\mathcal{E}_\gamma$ contains a term due to the vortex length (first integral) and one due to rotation (oriented integral since $dz = \mathbf{dl}\cdot e_z$). The rotation term becomes significant when Ω is of order $|\log\varepsilon|$. This derivation is justified using Γ-convergence techniques [85] and the introduction of currents. The convergence of Jacobians implies regularity of the currents and the fact that vortices are indeed described by regular curves.

The next step is to study the minimizing curves of $\mathcal{E}_\gamma$ according to Ω: if for some values of Ω, there are curves γ such that $\mathcal{E}_\gamma < 0$, then a vortex is favorable in the system. We find that indeed if β, which is related to the elongation of the condensate in (1.3), is small, then the minimizing curve is not along the z axis, but bending. We also analyze the properties of the local minimizers of $\mathcal{E}_\gamma$ and relate them to experimental observations and numerical computations illustrated in Figure 1.5: there are other types of vortex lines than the bent vortex, namely S vortices, which are only local minimizers of the energy.

In the case of several vortex lines, we derive an interaction energy between the lines, which is of lower order:

$$I(\gamma_i,\gamma_k) = \pi\int_{\gamma_i} \rho_{\mathrm{TF}} \log(\mathrm{dist}(x,\gamma_k))\, dl.$$

Critical velocities for the existence of n vortices

For the simplified problem where $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - r^2$ and $\mathcal{D}$ given by (1.3) is a disc in $\mathbf{R}^2$, we describe in Chapter 3 more detailed results about the minimizer $u \in H_0^1(\mathcal{D},\mathbf{C})$ of (1.8). We make an asymptotic expansion of the energy E_ε in terms of ε, which allows us to characterize the critical value for which the minimizer exhibits n vortices and get information on the location of these vortices. This is in the spirit of the techniques of Ginzburg–Landau vortices introduced by Bethuel–Brezis–Helein [32] in 2D, developed by Sandier–Serfaty [138], and used for this problem in $\mathbf{R}^2$ by Ignat–Millot [80, 81].

When the minimizer is vortex-free, $|u|^2 \approx \rho_{\mathrm{TF}}$. Vortices are points where u vanishes and around which there is a circulation. They are identified through small balls of characteristic size ε. Except for these balls, which are local perturbations of the wave function, $|u|$ is close to the vortex-free solution. The contribution of vortices to the energy is through their phase. If u is a minimizer with n vortices, then we will prove that

$$\begin{aligned} E_\varepsilon(u) = \mathcal{E}(\varepsilon) + \Big(\pi\rho_0 n|\log\varepsilon| - \frac{\pi\Omega}{2}n\rho_0^2\Big) \\ + \frac{\pi}{2}n\rho_0(n-1)\log\Omega + \min_{\mathbf{R}^{2n}} w + C_n + o(1), \end{aligned} \tag{1.11}$$

where $\mathcal{E}(\varepsilon)$ is the leading order term in ε representing the energy of the vortex-free solution, C_n is an explicit constant which depends only on n, $o(1)$ is a term which tends to 0 as ε tends to 0, and

$$w(b_1, \dots, b_n) = -\pi\rho_0 \sum_{i \neq j} \log|b_i - b_j| + \frac{\pi\rho_0}{2} \sum_i |b_i|^2. \tag{1.12}$$

A solution with vortices is energetically favorable as soon as the second term in the expansion becomes negative, that is, $\Omega\rho_0 > 2|\log\varepsilon|$. This provides the value of the critical velocity for the existence of vortices: if $\lim_{\varepsilon\to 0} \Omega/|\log\varepsilon| < 2/\rho_0$, then the minimizer does not have vortices on any compact subset of $\mathcal{D}$ as soon as ε is small enough. If $\lim_{\varepsilon\to 0} \Omega/|\log\varepsilon| < 2/\rho_0$, one has to go further into the expansion of E_ε to understand the number and location of vortices. Namely, the number of vortices is determined by

$$\omega_1 = \lim_{\varepsilon\to 0} \frac{\Omega - 2|\log\varepsilon|/\rho_0}{\log|\log\varepsilon|}. \tag{1.13}$$

If $n - 1 < \omega_1\rho_0 < n$, we prove that the minimizer has n vortices p_i. The rescaled location of the vortices $\tilde{p}_i = p_i/\sqrt{\Omega}$ tends to minimize the reduced energy w. One can check that the minimization of w indeed leads to patterns similar to those in Figure 1.3. In our scaling, the points p_i get close to the origin (at distance of order $1/\sqrt{\Omega}$ or $1/\sqrt{|\log\varepsilon|}$), but in initial physical units, this has to be rescaled by the size of the condensate, of order $1/\sqrt{\varepsilon}$.

If $\lim_{\varepsilon\to 0} \Omega/|\log\varepsilon| > 2/\rho_0$ but stays finite, the density of vortices is uniform in the system, but their interdistance remains much bigger than their characteristic size. Another regime, in which Ω becomes of order $1/\varepsilon^2$, will be addressed below. There, the interdistance between vortices becomes of order the vortex size.

Other trapping potentials

In recent ENS experiments [40, 150], an extra laser beam is added to the system and thus modifies the trapping potential. We will address this case in Chapter 4 for a two-dimensional condensate which can be described by replacing $\rho_{\mathrm{TF}} = \rho_0 - r^2$ by $\rho_0 - \alpha r^2 + \beta r^4$. According to the values of α and β, the region $\mathcal{D}$ where $\rho_{\mathrm{TF}} > 0$ changes topology and in particular, can be an annulus. In this case, we prove that the minimizer displays a giant vortex and determine the critical velocities Ω_d for which the circulation of this giant vortex is d. These critical velocities are of order 1, contrary to the case of the disc: if

$$\Lambda_1 d \leq \Omega < \Lambda_1(d+1) \text{ with } \Lambda_1 = \int_{\mathcal{D}} \frac{\rho_{\mathrm{TF}}(r)}{r^2},$$

then we prove that the minimizer has a degree d on any circle included in $\mathcal{D}$.

For velocities of order $|\log\varepsilon|$, the pattern of vortices changes. There is a giant vortex in the center with circulation proportional to $|\log\varepsilon|$, but there are also vortices

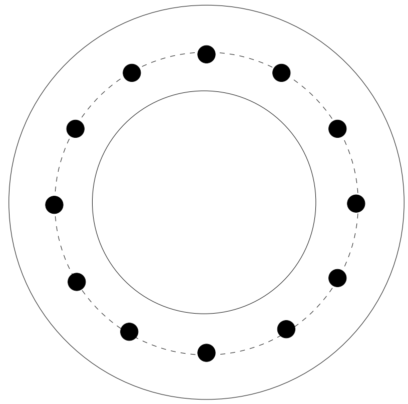


Fig. 1.6. Giant vortex with circle of vortices.

in the bulk, that is, inside the annulus, as illustrated in Figure 1.6: they are arranged on a specific circle that we are able to identify as the location of the maximum of the function $\xi(r)/\rho_{\mathrm{TF}}(r)$, where

$$\xi(r) := \int_r^{R_0} \rho_{\mathrm{TF}}(s)\left(s - \frac{1}{s\Lambda_1}\right) ds. \tag{1.14}$$

1.2.3 Remarks on the original problem

The original problem coming from the experiments is posed in the whole space. That is, one has to look for minimizers of

$$E_\varepsilon(u) = \int_{\mathbf{R}^n} \frac{1}{2}|\nabla u|^2 - \mathbf{\Omega} \times \mathbf{r} \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2}\left((|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}))^2 - (\rho_{\mathrm{TF}}(\mathbf{r})_-)^2\right) \tag{1.15}$$

under $\int_{\mathbf{R}^n} |u|^2 = 1$, where n =2 or 3 and $(\rho_{\mathrm{TF}})_-$ is the negative part of ρ_{TF}. Outside the domain $\mathcal{D}$, one can prove that the modulus of the minimizer decreases exponentially fast to zero. In fact, $|u|^2$ is close to the positive part of ρ_{TF} except in the vortex balls, which are small local perturbations of the density, and close to the boundary of $\mathcal{D}$. Our reduction to the bounded domain $\mathcal{D}$ removes the L^2 constraint but prescribes the value of ρ_0 through $\int_{\mathcal{D}} \rho_{\mathrm{TF}} = 1$. The boundary-layer thickness where $|u|^2$ matches ρ_{TF} is of order $\varepsilon^{2/3}$, and inside this boundary layer, u can be approximated by a solution of a Painlevé-type equation. We rescale the solution through $u(x, y, z) = \varepsilon^{1/3}\psi(\tilde{x}, \tilde{y}, \tilde{z})$, where $\tilde{x} = (\sqrt{\rho_0} - x)\varepsilon^{2/3}$, $y = \tilde{y}\varepsilon^{2/3}$, and $z = \tilde{z}\varepsilon^{2/3}$. This blows up the boundary of the cloud near $x = \sqrt{\rho_0}$, and ψ is almost a function of $\tilde{x}$ and a solution of the Painlevé equation [61, 54]

$$p'' + (2s\sqrt{\rho_0} - p^2)p = 0, \;\; p(s) \underset{s\to-\infty}{\to} 0, \;\; p(s) \underset{s\to\infty}{\sim} \sqrt{2\sqrt{\rho_0}s}. \tag{1.16}$$

As s tends to $-\infty$, the cubic term in the equation can be neglected and the asymptotic behaviour is given by an Airy function: $Ae^{-2(-s)^{2/3}/3}/(2(-s)^{1/4})$, where the value of A can be determined by a numerical integration. In the opposite limit at $+\infty$, the square root matches the local behaviour of $\sqrt{\rho_{\mathrm{TF}}}$ near the boundary of the cloud. The rigorous proof of the derivation of the Painlevé equation is still open. The role of this layer is important to understand the superfluid flow around an obstacle. It also has a leading-order contribution in the term $\mathcal{E}(\varepsilon)$ in the energy, though this does not influence the description of vortices.

1.2.4 Mean-field quantum Hall regime

In Chapter 5, we address a regime which is very different from the Thomas–Fermi regime (or small ε case), namely the fast rotation regime, when the rotational velocity $\tilde{\Omega}$ gets close to the trapping frequency ω_x. The small parameter will now be related to how close $\Omega = \tilde{\Omega}/\omega_x$ is to 1. The vortex lattice is dense and the characteristic size between vortices becomes of the same order as their interdistance. We restrict our analysis to a two-dimensional gas in the xy plane, assuming a strong confinement in the z axis, as described above, and we set $\alpha = 1$ for simplicity. The energy (1.2), taking into account the 2D reduction can be written

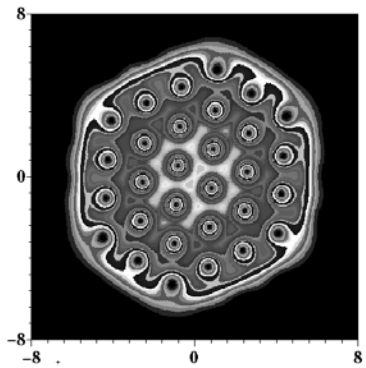
$$E(\psi) = \int_{\mathbf{R}^2} \frac{1}{2}|\nabla\psi - i\boldsymbol{\Omega} \times \mathbf{r}\psi|^2 + \frac{1}{2}(1-\Omega^2)r^2|\psi|^2 + \frac{1}{2}Na|\psi|^4, \qquad (1.17)$$

under $\int_{\mathbf{R}^2} |\psi|^2 = 1$. The term a is the effective scattering length and takes into account the 2D reduction. The rescaled rotational velocity is along the z axis $\boldsymbol{\Omega} = \Omega e_z$, and $\mathbf{r} = (x, y)$. Note that we have written the first two terms of (1.2) as the expansion of a complete square, and thus subtracted the extra term. In order for the energy to be bounded below, we need to have $\Omega < 1$, which means that the trapping potential remains stronger than the rotating force. Our limiting regime is when Ω tends to 1. The minimization is performed in $\mathbf{R}^2$ and not just in a bounded domain, because the size of the condensate increases as Ω approaches 1: the characteristic size of the condensate, that is, the region where the wave function is significant, is proportional to $(1-\Omega)^{-1/4}$. In this region, vortices are arranged on a triangular lattice, while outside, the wave function is of small amplitude, yet the analysis of the zeroes is still of interest. The amplitude of the wave function and the location of the zeroes are plotted in Figure 1.7.

This regime displays a strong analogy with quantum Hall physics: the first part of the Gross–Pitaevskii energy is similar to that of a charged particle in a uniform magnetic field. Therefore, the ground state becomes degenerate as in the case of the Landau levels obtained for the motion of a charge in a magnetic field.

Lowest Landau level

The first term in the energy is identical to the energy of a particle placed in a uniform magnetic field $2\boldsymbol{\Omega}$. The minimizers for



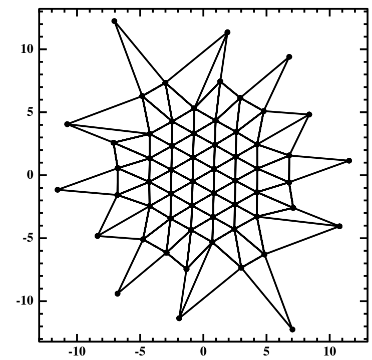


Fig. 1.7. An example of (right): a configuration of zeroes minimizing the energy for $\Omega = 0.999$, $Na = 3$. (left): density plot of $|\psi|$.

$$\int_{\mathbf{R}^2} \frac{1}{2} |\nabla \psi - i\mathbf{\Omega} \times \mathbf{r}\psi|^2 \text{ under } \int_{\mathbf{R}^2} |\psi|^2 = 1. \tag{1.18}$$

are well known [104] through the study of the eigenvalues of the operator $-(\nabla - i\mathbf{\Omega} \times \mathbf{r})^2$. The minimum is Ω and the corresponding eigenspace is of infinite dimension and called the lowest Landau level (LLL). This can be studied using a change of gauge and a Fourier transform in one direction. The other eigenvalues are $(2k+1)\Omega$, $k \in \mathbf{N}$. A basis of the first eigenspace is given by

$$\psi(x, y) = P(z)e^{-\Omega|z|^2/2} \text{ with } z = x + iy, \tag{1.19}$$

where P varies in a basis of polynomials. The closure of this space in $L^2(\mathbf{R}^2)$ is made up of functions of the type (1.19), where P varies in the space of holomorphic functions. In this framework, vortices are the zeroes of the polynomial or holomorphic function and are thus easy to identify.

Our aim is to restrict the minimization of the energy (1.17) to this eigenspace. We will see that as Ω approaches 1, the second and third terms in the energy (1.17) produce a contribution of order $\sqrt{1 - \Omega}$, which is much smaller than the gap between two eigenvalues of $-(\nabla - i\Omega \times \mathbf{r})^2$, namely 2Ω. Thus, this restriction is natural as a first step, but we are not able to provide a full rigorous justification. When ψ is restricted to the lowest Landau level (1.19), the energy (1.17) reads

$$E(\psi) = \Omega + E_{\mathrm{LLL}}(\psi) := \Omega + \int_{\mathbf{R}^2} \frac{(1 - \Omega^2)}{2} r^2 |\psi|^2 + \frac{Na}{2} |\psi|^4. \tag{1.20}$$

Expected shape of the minimizer

We want to minimize the energy (1.20) under $\int_{\mathbf{R}^2} |\psi|^2 = 1$, when ψ is given by (1.19). A numerical study consists in writing $P(z) = p_0 \prod_{i=1}^n (z - z_i)$ and minimizing the energy on the location z_i of the zeroes and the degree n. The result is illustrated in Figure 1.7: there are 30 vortices in the left picture, where $|\psi|$ is plotted, and 56 vortices in total in the right view. Our numerical observations indicate that vortices are located on a regular triangular lattice in the central region (about 30 vortices), while the lattice is distorted towards the edges, in the region of very low density where $|\psi|$ is not visible. The optimal degree seems proportional to $1/\sqrt{1-\Omega}$. Our aim is to justify these observations rigorously. We are going to construct an upper bound, which is close to the numerical observations. We will use homogenization techniques and double-scale convergence. The lower bound is still an open question.

An important issue is to understand that the existence of zeroes of P far away modifies its decay. We assume that the zeroes lie on a lattice $\ell = \alpha(\mathbf{Z} + \tau\mathbf{Z})$, with $\alpha \in \mathbf{R}^+$ and $\tau \in \mathbf{C}$, of unit cell with volume V. Let $P_R(z) = p_0 \prod_{|z_i|<R,\ z_i \in \ell}(z - z_i)$. Then as R tends to infinity,

$$\left| P_R(z) e^{-\Omega|z|^2/2} \right| \stackrel{*}{\rightharpoonup} \psi(z) := \eta(z) e^{-|z|^2/\sigma^2} \text{ with } \frac{1}{\sigma^2} = \Omega - \frac{\pi}{V}, \tag{1.21}$$

where η is a periodic function on the lattice which vanishes in each cell. The decay of the wave function is thus modified by the volume of the cell through σ. In fact, the function η is explicit and minimizes the ratio

$$b = \frac{⨍ |\eta|^4}{(⨍ |\eta|^2)^2}$$

(where $⨍$ denotes the integral on a cell per unit volume) among all periodic functions on a lattice which vanish once in each cell. The lattice minimizing the ratio b among all possible lattices is the triangular or hexagonal one, that is, $\tau = e^{2i\pi/3}$. This is, the parameter that appears for type-II superconductors for what is called the Abrikosov lattice [2, 93].

If the volume of the cell tends to π, then the difference of scales in (1.21) between η and the Gaussian allows a separation in the energy, and we obtain

$$E_{\mathrm{LLL}}(\psi) \underset{\sigma\to\infty}{\sim} (1-\Omega)\sigma^2 + \frac{Nab}{4\pi\sigma^2}.$$

This expression is minimal when the two terms are equal and σ^4 is thus proportional to $1/(1-\Omega)$, which provides the upper bound for the energy

$$\sqrt{\frac{Nab(1-\Omega)}{\pi}}. \tag{1.22}$$

Let us emphasize the presence of the coefficient b: it corresponds to the contribution of the vortex lattice to the energy.

In fact, we prove more: any type of slowly varying profile (and not only a Gaussian) can be approximated as Ω tends to 1, using this ansatz by distorting the lattice. We are able to relate the distortion of the lattice to the decay of the wave function. This is how we construct our upper bound. The search for the optimal slowly varying profile is motivated by very recent physics papers: in a seminal paper, Ho [79] computed the energy (1.20) of a configuration of type (1.19), where the z_i are located on a triangular lattice, and found that the wave function averaged over vortex cells has a Gaussian decay. This was confirmed by [27]. The issue of understanding the properties of the vortex lattice and the decay of the wave function is a challenging one: only recently did Cooper, Komineas, and Read [49] observe numerically the distortion of the lattice on the edges of the condensate (similar to Figure 1.7) and the decay of the wave function, which is closer to an inverted parabola than a Gaussian.

Let us motivate the inverted parabola: a rough analysis of a lower bound for (1.20) under $\int_{\mathbf{R}^2} |\psi|^2 = 1$ without the constraint of being in the space (1.19) implies that the minimizer is the inverted parabola

$$|\psi|^2(z) = \frac{2}{\pi R^2}\left(1 - \frac{|z|^2}{R^2}\right) 1_{\{|z| \le R\}}, \;\; R = \left(\frac{2Na}{\pi(1-\Omega)}\right)^{1/4}. \tag{1.23}$$

The energy of such a test function is $2\sqrt{2}\sqrt{Na(1-\Omega)/\pi}/3$, without coefficient b. The restriction to $f = \psi e^{\Omega|z|^2/2}$ being holomorphic prevents one from achieving this specific inverted parabola. But a distortion of the vortex lattice inside the space (1.19) provides a weak-star approximation of the inverted parabola and will modify the radius R by a coefficient $b^{1/4}$ coming from the contribution of the lattice to the energy through the function η. The distortion of the lattice on the edges improves the upper bound (1.22) and yields

$$\frac{2\sqrt{2}}{3}\sqrt{\frac{Nab(1-\Omega)}{\pi}}.$$

The lower bound given by the inverted parabola has a difference of a factor $\sqrt{b}$ due to the lattice contribution. The proof that the lower bound should include $\sqrt{b}$ is still open.

Additional results can be obtained using the explicit expression of the projector onto the space spanned by (1.19): for any $g(z, \bar{z})$,

$$\Pi(g) = \frac{\Omega}{\pi}\int_{\mathbf{R}^2} e^{\Omega z\overline{z'}} e^{-\Omega|z'|^2} g(z', \bar{z}'). \tag{1.24}$$

This provides in particular an equation satisfied by the minimizer. This allows us to prove that the minimizer has an infinite number of zeroes, and thus is not achieved by a polynomial. We would like to get more information on the distribution of vortices for the minimizer.

1.2.5 Flow around an obstacle

Finally, in Chapter 7 we address another experiment with Bose–Einstein condensates which exhibits vortices [128] (see also [67, 117, 129]). It consists in a superfluid flow around an obstacle: at small velocity, there are no vortices in the system, the flow is indeed superfluid, and we prove the existence of stationary solutions. When the velocity increases, vortices are nucleated and the flow becomes time dependent. We show numerical simulations and open problems related to this regime. The mathematical difficulty is thus to get existence results for equations of the type

$$\Delta\psi - 2ic\partial_x\psi + \psi(a(x, y, z) - |\psi|^2) = 0$$

when the velocity of the flow c is small, in an exterior domain, which is the exterior of the obstacle. The precise definition of a (depending on the trapping potential) and the domain will be given in the course of the chapter. They are related to the Painlevé boundary layer, where the nucleation of vortices takes place.

2

The Mathematical Setting: A Survey of the Main Theorems

This book contains results in three directions: the Thomas–Fermi regime or small ε problem, where there is a bounded number of vortices in the system (Chapters 3, 4, 6); the fast-rotation regime, which displays a vortex lattice (Chapter 5); and the experiment of a superfluid flow around an obstacle (Chapter 7). The tools and techniques needed to address these problems are very different: energy expansion using a small parameter for the Thomas–Fermi regime; double–scale convergence, homogenization techniques, and Fock–Bargmann space for the fast–rotating regime; energy estimates and nondegeneracy of a solution for the superfluid flow. The main mathematical results are summarized in the present chapter.

2.1 Small ε problem

A large part of this book is concerned with the study of minimizers $u \in H_0^1(\mathcal{D}; \mathbf{C})$ of the following energy, where $\mathcal{D}$ is a bounded domain in $\mathbf{R}^2$ or $\mathbf{R}^3$:

$$E_\varepsilon(u) = \int_{\mathcal{D}} \frac{1}{2}|\nabla u|^2 - \frac{1}{2}\mathbf{\Omega}_\varepsilon \times \mathbf{r} \cdot (iu\nabla\bar{u} - i\bar{u}\nabla u) + \frac{1}{4\varepsilon^2}(|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}))^2, \quad (2.1)$$

where ε is a small parameter, $\mathbf{\Omega}_\varepsilon = \Omega_\varepsilon e_z$ is a given vector parallel to the z direction, $\mathbf{r} = (x, y, z)$, $\mathbf{\Omega}_\varepsilon \times \mathbf{r} = (-\Omega_\varepsilon y, \Omega_\varepsilon x, 0)$, $\bar{u}$ is the complex conjugate of u, and $\rho_{\mathrm{TF}}(\mathbf{r})$ is a regular prescribed function modelling a trapping potential. In the following, we will use the notation $(iu, \nabla u)$ of the scalar product in $\mathbf{C}$, to denote the term $(iu\nabla\bar{u} - i\bar{u}\nabla u)/2$. The domain $\mathcal{D}$ is defined as

$$\mathcal{D} = \{\rho_{\mathrm{TF}}(\mathbf{r}) > 0\} \quad (2.2)$$

and the function ρ_{TF} is normalized in such a way that $\int_{\mathcal{D}} \rho_{\mathrm{TF}} = 1$. The original motivation coming from experiments corresponds to $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - (x^2 + \alpha^2 y^2 + \beta^2 z^2)$, where α is close to 1 and β is small. In order to understand the main difficulties of the problem, we will start with a model case corresponding to a problem in $\mathbf{R}^2$ (formally $\beta = 0$), where in fact $\mathbf{r} = (x, y)$, $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - (x^2 + y^2)$, and $\mathcal{D}$ is thus a disc in $\mathbf{R}^2$. Then, we will focus on other types of functions ρ_{TF} for $\mathbf{r}$ in $\mathbf{R}^2$ and finally address the original case of the experiments for the harmonic potential in $\mathbf{R}^3$.

The main issue is to analyze, according to the value of Ω_ε, the properties of the minimizers u when ε is small and determine the location of the zero set of u. This will rely on an asymptotic expansion of the energy, and techniques introduced for Ginzburg–Landau-type problems by Bethuel–Brezis–Helein [32] in 2D and Riviere [131] in 3D, and then developed and extended by Jerrard [84], Jerrard–Soner [86], and Sandier–Serfaty [138]. Vortices will be identified as balls in 2D or tubes in 3D of characteristic size ε, containing the zero set of u around which the phase changes.

2.1.1 The two-dimensional setting

We will first address the case in which $\mathbf{r}$ lies in $\mathbf{R}^2$, and $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_{\mathrm{TF}}(x, y)$, so that $\mathcal{D}$ is a two-dimensional domain.

The model case

In Chapter 3, we will study the minimizers $u \in H_0^1(\mathcal{D}, \mathbf{C})$ of (2.1) when $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - r^2$ and $\mathcal{D}$ given by (2.2) is the disc in $\mathbf{R}^2$ of radius $\sqrt{\rho_0}$. We will make an asymptotic expansion of the energy E_ε in terms of ε, which will allow us to characterize the critical value for which the minimizer exhibits n vortices and the location of these vortices. This is in the spirit of the techniques of Ginzburg–Landau vortices introduced by Bethuel–Brezis–Helein [32] in 2D, developed by Serfaty [143, 144] and Sandier–Serfaty [138], and used for this problem in $\mathbf{R}^2$ by Ignat–Millot [80, 81]. If u is a minimizer with n vortices, then we will prove that as ε tends to 0,

$$\begin{aligned} E_\varepsilon(u) = \mathcal{E}(\varepsilon) + \left(\pi\rho_0 n|\log\varepsilon| - \frac{\pi\Omega_\varepsilon}{2} n\rho_0^2\right) \\ + \frac{\pi}{2} n\rho_0(n-1)\log\Omega_\varepsilon + \min_{\mathbf{R}^{2n}} w + C_n + o(1), \end{aligned} \tag{2.3}$$

where $\mathcal{E}(\varepsilon)$ is the leading-order term in ε representing the energy of the vortex-free solution, C_n is an explicit constant which depends only on n, $o(1)$ is a term which tends to 0 as ε tends to 0, and

$$w(b_1, \dots, b_n) = -\pi\rho_0 \sum_{i\neq j} \log|b_i - b_j| + \frac{\pi\rho_0}{2}\sum_i |b_i|^2. \tag{2.4}$$

A solution with vortices is energetically favorable as soon as the second term in the expansion becomes negative, that is, $\Omega_\varepsilon\rho_0 > 2|\log\varepsilon|$. This provides the value of the critical velocity for the existence of vortices, and we prove that for Ω_ε smaller than this critical velocity, the minimizer does not exhibit vortices:

Theorem 2.1. *Let u_ε be a sequence of minimizers of E_ε in $H_0^1(\mathcal{D})$ and assume that Ω_ε depends on ε in such a way that*

$$\lim_{\varepsilon \to 0} \frac{\Omega_\varepsilon}{|\log \varepsilon|} = \omega_0. \tag{2.5}$$

Then $\omega_0^* = 2/\rho_0$ *is a critical value in the sense that, if* $\omega_0 < \omega_0^*$ *for any compact subset* K *of* $\mathcal{D}$*, if* ε *is smaller than some* ε_K*, then* u_ε *does not vanish in* K*. In addition, as* ε *tends to* 0*,* $|u_\varepsilon|$ *converges to* $\sqrt{\rho_{\mathrm{TF}}}$ *in* $L^\infty_{\mathrm{loc}}(\mathcal{D})$*, and*

$$E_\varepsilon(u_\varepsilon) = \mathcal{E}(\varepsilon) + o(1), \tag{2.6}$$

where $\mathcal{E}(\varepsilon)$ *does not depend on* u_ε *or* Ω_ε*.*

If ω_0 reaches the critical value ω_0^*, then the number of vortices in the system depends on the next term in the expansion of Ω_ε. The critical velocity for the existence of vortices is determined by the fact that the leading term in the expansion (2.3) is the same for n and $n+1$ vortices:

Theorem 2.2. *We assume a specific asymptotic form for the rotation* Ω*,*

$$\Omega_\varepsilon = \omega_0^* |\log \varepsilon| + \omega_1 \log|\log \varepsilon|. \tag{2.7}$$

Let u_ε *be a sequence of minimizers of* E_ε *in* $H^1_0(\mathcal{D})$*:*

(i) If $\omega_1 < 0$*, then the conclusion of Theorem 2.1 holds.*

(ii) If $\omega_1^n < \omega_1 < \omega_1^{n+1}$*, with* $\omega_1^n = 2(n-1)/\rho_0$*, for any compact subset* K *of* $\mathcal{D}$ *containing a neighborhood of the origin, if* ε *is smaller than some* ε_K*,* u_ε *has exactly n vortices* p_i^ε *of degree one in* K*. Moreover,*

$$|p_i^\varepsilon| < C/\sqrt{\Omega_\varepsilon} \text{ for any } i \text{ and } |p_i^\varepsilon - p_j^\varepsilon| > C/\sqrt{\Omega_\varepsilon},$$

where C *is independent of* ε*. Let* $\tilde{p}_i^\varepsilon = p_i^\varepsilon/\sqrt{\Omega_\varepsilon}$*. Then the configuration* $\tilde{p}_i^\varepsilon$ *tends to minimize the energy* w *defined in* $\mathbf{R}^{2n}$ *by (2.4).*

The proof relies on the splitting of the energy introduced by Lassoued–Mironescu [97], and a construction of an upper bound and a lower bound which provide the energy expansion (2.3). The construction of the lower bound is made in two steps: the first one provides an upper bound of the number of vortices (as in [136]) according to the bound on the rotational velocity, that is, the bound on ω_1. The second step uses the techniques of bad discs introduced by Bethuel, Brezis, and Helein [32] and allows one to localize the vortices and yields the renormalized energy w.

Let η_ε be the minimizer of E_ε with $\Omega_\varepsilon = 0$. Then the energy decouples into the profile part η_ε, the contribution of vortices, and the contribution of rotation:

$$E_\varepsilon(u) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_{\eta_\varepsilon}(v), \text{ where } \mathcal{E}_{\eta_\varepsilon}(v) = G_{\eta_\varepsilon}(v) + L_{\eta_\varepsilon}(v), \tag{2.8}$$

$$\text{and} \quad G_{\eta_\varepsilon}(v) = \int_{\mathcal{D}} \frac{\eta_\varepsilon^2}{2}|\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2, \tag{2.9}$$

$$L_{\eta_\varepsilon}(v) = -\int_{\mathcal{D}} \eta_\varepsilon^2 \Omega \mathbf{r}^\perp \cdot (iv, \nabla v), \tag{2.10}$$

where $\mathbf{r}^\perp = (-y, x)$. The term G_{η_ε} is very similar to the energy studied in [32], with the addition of a weight η_ε^2 which is very close to ρ_{TF}. This part of the energy allows us to define the vortex structure and provides the $|\log \varepsilon|$ term in the expansion, as well as the renormalized energy. The rotation term provides the negative quantity $-\pi \Omega_\varepsilon \sum_i \rho_{\mathrm{TF}}^2(p_i^\varepsilon)/2$. The vortices which contribute to the energy expansion are located close to the origin (at distance of order $1/\sqrt{|\log \varepsilon|}$) and thus the weight in the expansion becomes the value of ρ_{TF} at zero, that is, ρ_0. The presence of ρ_{TF}^2 in the rotation term is a very special feature of the harmonic potential: it corresponds to the fact that a primitive of $r\rho_{\mathrm{TF}}(r)$ for our special choice of ρ_{TF} is $-\rho_{\mathrm{TF}}^2/4$.

Since the weight ρ_{TF} vanishes on the boundary of $\mathcal{D}$, we get information only on compact subsets of $\mathcal{D}$ and are not able to analyze vortices up to the boundary. This is the topic of interesting open issues.

Other trapping potentials

The analysis is carried on in Chapter 4 with other functions $\rho_{\mathrm{TF}}(r)$ when $\mathbf{r} = (x, y)$ still lies in $\mathbf{R}^2$ and $\mathcal{D}$ given by (2.2) is a domain in $\mathbf{R}^2$. We will address the case of a nonradial function with a harmonic growth and a radial function whose support is an annulus.

The case of the experiments is a harmonic potential with an inhomogeneity in the x and y directions, namely $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - x^2 - \alpha^2 y^2$, with $\alpha \neq 1$. The results are very similar to those of the previous section, except that now the energy expansion and the critical velocity depend on α. This has been addressed by Ignat–Millot [80, 81]. The main difference with the case $\alpha = 1$ lies in the fact that the vortex-free solution, that is, the minimizer η_ε of E_ε under the condition that it does not vanish in $\mathcal{D}$, is no longer a real-valued function but has a globally defined phase S_ε. Its modulus is very close to ρ_{TF}. Due to the specific growth of ρ_{TF}, as ε tends to 0, this phase converges to S, which is proportional to $(\alpha^2 - 1)xy$. If ρ_{TF} had another type of inhomogeneity in x and y, this limit would not be so easy to identify and the expansion of energy not so precise. In the splitting of energy, we use $|\eta_\varepsilon| e^{iS}$ as a comparison function and prove that (see Chapter 4, Theorem 4.1, for a precise statement)

$$\begin{aligned} E_\varepsilon(u) = E_\varepsilon(|\tilde{\eta}_\varepsilon| e^{iS}) + \Big(\pi\rho_0 n |\log \varepsilon| - \frac{\pi \Omega_\varepsilon}{1+\alpha^2} n \rho_0^2\Big) \\ + \frac{\pi}{2} n \rho_0 (n-1) \log \Omega_\varepsilon + \min_{\mathbf{R}^{2n}} w + C_{n,\alpha} + o(1), \end{aligned} \tag{2.11}$$

where

$$w(b_1, \dots, b_n) = -\pi\rho_0 \sum_{i \neq j} \log |b_i - b_j| + \frac{\pi \rho_0}{1+\alpha^2} \sum_i |b_i|_\alpha^2 \tag{2.12}$$

and $|\mathbf{r}|_\alpha^2 = x^2 + \alpha^2 y^2$. The critical velocities will now depend on α, as well as the renormalized energy w, through the weighted norm $|\mathbf{r}|_\alpha^2$.

We also deal with a radial function ρ_{TF} with support in an annulus such as $\rho_0 + (b-1)r^2 - (k/4)r^4$ with $b > 1 + (3k^2/4)^{1/3}$. We refer to $B_{R_0} \setminus \overline{B}_{R_1}$ as the domain $\mathcal{D}$. The special feature of this trapping potential is that vortices appear as giant vortices: there is a degree in the annulus, but the vortices are in the inner disc. It is very difficult both experimentally and mathematically to discriminate between a giant vortex centered at the origin and several vortices of degree one, located close to the origin in the region where the wave function is small. For this reason, we only provide information about the circulation in the annulus, which has the effect of a giant vortex. The critical velocity to nucleate a vortex is now of order 1, contrary to the previous cases, where it was of order $|\log \varepsilon|$:

Theorem 2.3. *Let $g_0(d) = \Lambda_1 d^2/2 - \Omega d$ for d in $\mathbf{Z}$,where $\Lambda_1 = \int_{\mathcal{D}} \rho_{\mathrm{TF}}(r)/r^2$. Let $\Omega_d = \Lambda_1(d - 1/2)$ for $d \geq 1$ and $\Omega_0 = 0$. If u_ε is a sequence of minimizers of E_ε, and $\Omega_d \leq \Omega_\varepsilon < \Omega_{d+1}$, then:*

(i) $E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon) \to g_0(d)$, as ε tends to 0.
(ii) There exists a subsequence $\varepsilon \to 0$ and $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ such that

$$\frac{u_\varepsilon}{\eta_\varepsilon} \to \alpha e^{id\theta} \quad \text{in } H^1_{\mathrm{loc}}(\mathcal{D}), \text{ and } \quad \left|\frac{u_\varepsilon}{\eta_\varepsilon}\right| \to 1, \quad \text{locally uniformly in } \mathcal{D}.$$

(iii) For every fixed r such that $\partial B_r(0) \subset \mathcal{D}$, $\deg(\frac{u_\varepsilon}{\eta_\varepsilon}, \partial B_r) = d$ for ε sufficiently small.

For velocities Ω_ε of order $|\log \varepsilon|$, the pattern is quite different: vortices exist in the annulus and they are arranged regularly on a specific circle as illustrated in Figure 1.6. We prove that the degree of the giant vortex is proportional to $|\log \varepsilon|$, and we characterize the circle where vortices appear in the annulus as the location of the maximum of the function $\xi(r)/\rho_{\mathrm{TF}}(r)$, where

$$\xi(r) := \int_r^{R_0} \rho_{\mathrm{TF}}(s) \left(s - \frac{1}{s\Lambda_1}\right) ds. \tag{2.13}$$

The precise statement is given in Theorem 4.4.

2.1.2 The three-dimensional setting

We now address the case when $\mathcal{D}$ is an ellipsoid in $\mathbf{R}^3$. In Chapter 6, we study the minimizers $u \in H_0^1(\mathcal{D}, \mathbf{C})$ of (2.1) when $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - (x^2 + \alpha^2 y^2 + \beta^2 z^2)$ and $\mathcal{D}$ is given by (2.2). We want to make an asymptotic expansion of E_ε as in the previous cases, but the three-dimensional setting does not allow us to get as many terms in the expansion. Our mathematical results deal mainly with the single-vortex solution and are aimed at proving the bending property. We will see that the shape of vortices depends on the elongation β in the z direction. We are able to determine the critical velocity for which vortices are favorable, and their optimal shape.

In this derivation, vortices can be represented by oriented curves, which are Lipschitz functions $\gamma : (0, 1) \to \mathcal{D}$. We are going to prove that if the minimizer has a single singularity line γ, then the energy decouples into the energy of the vortex-free solution $\mathcal{E}_0(\varepsilon)$ and a contribution from the vortex line γ, that is, as ε tends to 0,

$$E_\varepsilon(u) = \mathcal{E}_0(\varepsilon) + \pi|\log\varepsilon|\mathcal{E}_\gamma + o(|\log\varepsilon|),$$

where

$$\mathcal{E}_\gamma = \int_\gamma \rho_{\mathrm{TF}}\, dl - \frac{\Omega_\varepsilon}{(1+\alpha^2)|\log\varepsilon|}\int_\gamma \rho_{\mathrm{TF}}^2\, dz. \tag{2.14}$$

The energy of the vortex line γ contains a term due to the vortex length (first integral) and one due to rotation (oriented integral since $dz = \mathbf{dl}\cdot e_z$), which tends to force the vortex to be parallel to the z axis, while the other term wants to minimize the length. This is why, according to the geometry of the trap, the shape of the vortex varies. The rotation term becomes significant when Ω_ε is of order $|\log\varepsilon|$. This derivation is justified using Γ-convergence techniques and the introduction of currents. Vortices are identified through the study of the Jacobians, namely

$$Jv = \sum_{j<k} v_{x_j} \wedge v_{x_k}. \tag{2.15}$$

The convergence of Jacobians to a limiting current is proved and allows us to get regularity on the limiting singularity line γ and define $\mathcal{E}_\gamma$.

The next step is to study the minimizing curves of $\mathcal{E}_\gamma$ according to Ω_ε. For fixed Ω, if there are curves γ such that $\mathcal{E}_\gamma < 0$, then a vortex is favorable in the system. We find that indeed if β, which is related to the elongation of the condensate in (1.3), is small, then the minimizing curve is not along the z axis, but bending. We also analyze the properties of the local minimizers of $\mathcal{E}_\gamma$ (existence of S vortices as in Figure 1.5) and relate them to experimental observations.

The main results can be summarized in the following theorem, which characterizes the critical velocity below which the minimizer is vortex-free:

Theorem 2.4. *Let* $\bar\Omega = \lim_{\varepsilon\to 0}\Omega_\varepsilon/(1+\alpha^2)|\log\varepsilon|$ *and*

$$\bar\Omega_1 = \inf\{\bar\Omega,\ \exists\gamma \text{ with } \mathcal{E}_\gamma < 0\}.$$

Then $1 < \bar\Omega_1\rho_0 < 5/4$. *For* $\bar\Omega < \bar\Omega_1$, *the global minimizer* u_ε *is asymptotically vortex-free in* $\mathcal{D}$. *For* $\bar\Omega > \bar\Omega_1$, *the minimizer has vortices. The straight vortex does not minimize the reduced energy* $\mathcal{E}_\gamma$ *if* $\beta < \sqrt{2/13}$.

2.2 Vortex lattice

The other main part of this book (Chapter 5) is dedicated to another regime, in which the rotational velocity gets large: the condensate size increases with the velocity, vortices are no longer small balls, and their distance is of the same order as their size. They are arranged on a triangular lattice, distorted towards the edges of the condensate.

We consider the minimization of

$$E_{\mathrm{LLL}}(\psi) = \int_{\mathbf{R}^2} \frac{(1-\Omega^2)}{2} r^2 |\psi|^2 + \frac{Na}{2}|\psi|^4 \text{ under } \int_{\mathbf{R}^2} |\psi|^2 = 1 \tag{2.16}$$

for functions $\psi(z) = f(z)e^{-\Omega|z|^2/2}$, where f is holomorphic. As before, we are interested in the number and location of vortices. In this setting, we expect a vortex lattice of characteristic spacing of order 1 and a condensate of characteristic size R such that R^4 is proportional to $1/(1-\Omega)$. The ball B_R is the region where the wave function is dominant. The results that we will present deal with an upper bound for the energy, the lower bound remaining an open issue.

Theorem 2.5. *Let ℓ be a lattice, Q its unit cell. Assume that $V = |Q| > \pi$. Let*

$$\psi_R(z) = A_R \prod_{j \in \ell \cap B_R} (z - j) e^{-\Omega|z|^2/2} \tag{2.17}$$

with A_R chosen such that $\|\psi_R\|_{L^2(\mathbf{R}^2)} = 1$. Then as R tends to ∞,

$$|\psi_R(z)| \overset{*}{\rightharpoonup} \psi(z) = \frac{1}{\sqrt{\pi}\sigma} \eta(z) e^{-|z|^2/(2\sigma^2)}, \tag{2.18}$$

where

$$\frac{1}{\sigma^2} = \Omega - \frac{\pi}{V} \tag{2.19}$$

and η is a periodic function which vanishes at each point of ℓ. In addition, $\lim_{R\to+\infty} E_{\mathrm{LLL}}(\psi_R) = E_{\mathrm{LLL}}(\psi)$. As σ tends to infinity, then

$$E_{\mathrm{LLL}}(\psi) \sim (1-\Omega)\sigma^2 + \frac{1}{4}\frac{Nab}{\pi\sigma^2}, \quad \text{where } b = \frac{⨍ |\eta|^4}{(⨍ |\eta|^2)^2}. \tag{2.20}$$

Here, $⨍$ denotes the integral on a cell per unit volume.

The main feature of the periodic lattice is to modify the decay of the Gaussian from $e^{-\Omega|z|^2/2}$ to $e^{-|z|^2/2\sigma^2}$, where σ depends on the volume through (2.19). We need to choose the optimal σ in (2.20), which yields

$$\sigma^4(1-\Omega) = \frac{1}{4}\frac{Nab}{\pi}. \tag{2.21}$$

This value of σ indeed satisfies $\sigma \to +\infty$ as Ω tends to 1. The estimate of the energy is thus

$$E_{\mathrm{LLL}}(\psi) \underset{\Omega\to 1}{\sim} \sqrt{\frac{Nab}{\pi}(1-\Omega)}. \tag{2.22}$$

Let us emphasize the presence of the coefficient b: it takes into account the averaged vortex contribution on each cell. As in the case of superconductors near H_{c_2}, for the Abrikosov lattice, the optimal lattice minimizing the ratio b is the hexagonal one [93], providing 1.16 as a value for b.

In fact, the function η is explicit and related to the Abrikosov problem [2, 93]

$$\eta(z) = e^{\Omega z^2/2} e^{-\Omega |z|^2/2} \Theta(az, e^{2i\pi/3}) \text{ where } a = \sqrt{\frac{\sqrt{3}}{2\pi}}, \tag{2.23}$$

$$\Theta(v, \tau) = \frac{1}{i} \sum_{n=-\infty}^{+\infty} (-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}, \quad v \in \mathbf{C}. \tag{2.24}$$

Given the invariance properties of the Theta function on a lattice, $|\eta|$ is periodic on the lattice $a\mathbf{Z} + ae^{2i\pi/3}\mathbf{Z}$. The function given by (2.23) minimizes the ratio

$$b = \frac{⨍ |\eta|^4}{(⨍ |\eta|^2)^2}$$

among all periodic functions on a lattice which vanish once in each cell.

We construct a test function that should be close to the minimizer. It has a triangular lattice in the region where the wave function is significant and is distorted outside. The main observation is that modifying the location of the vortices from a regular lattice can change the decay of the wave function and hence improve the energy estimate.

Theorem 2.6. *There exists a sequence of functions ψ_Ω with $\psi_\Omega e^{|z|^2/2}$ holomorphic, such that as Ω tends to* 1*,*

$$E_{\mathrm{LLL}}(\psi_\Omega) \sim \frac{2\sqrt{2}}{3} \sqrt{\frac{Nab}{\pi}(1-\Omega)}. \tag{2.25}$$

This estimate is indeed better than the one for the regular lattice (2.22). Let us justify what kind of slowly varying profile (better than the Gaussian) produces an improvement in the energy estimate: if one considers the minimization of $E_{\mathrm{LLL}}(\psi)$ without the holomorphic constraint on f, then the minimization process yields that $(1-\Omega^2)|z|^2/2 + Na|\psi|^2 - \mu = 0$, where μ is the Lagrange multiplier due to the constraint $\int |\psi|^2 = 1$, so that $|\psi|$ is the inverted parabola

$$\alpha^2(z) = \frac{2}{\pi R^2}\left(1 - \frac{|z|^2}{R^2}\right) 1_{\{|z| \le R\}}, \quad R = \sqrt{\mu} = \left(\frac{2Na}{\pi(1-\Omega)}\right)^{1/4}. \tag{2.26}$$

The energy of such a test function is $2\sqrt{2Na(1-\Omega)/\pi}/3$, that is, (2.25), but without the coefficient b. The restriction to $f = \psi e^{\Omega|z|^2/2}$ being holomorphic prevents us from achieving this specific inverted parabola. A distortion of the vortex lattice provides a weak-star approximation of the inverted parabola but will modify the radius R by a coefficient $b^{1/4}$ coming from the contribution of the lattice to the energy through the function η. This is why our upper bound contains b, and we believe that the lower bound should as well.

Another approach to this problem is to introduce the so-called Fock–Bargmann space [26, 110]

$$\mathcal{F} = \left\{ f \text{ is holomorphic }, \int_{\mathbf{R}^2} |f|^2 e^{-\Omega|z|^2} < \infty \right\}. \tag{2.27}$$

This space is a Hilbert space endowed with the scalar product $\langle f, g\rangle_{\mathcal{F}} = \int_{\mathbf{R}^2} f(z) g(z) e^{-\Omega|z|^2}$. The projection of a general function $g(z, \bar{z})$ onto $\mathcal{F}$ is explicit :

$$\Pi(g) = \frac{\Omega}{\pi} \int_{\mathbf{R}^2} e^{\Omega z \overline{z'}} e^{-\Omega|z'|^2} g(z', \bar{z}'). \tag{2.28}$$

If g is a holomorphic function, then an integration by parts yields $\Pi(g) = g$. Using this expression, we are able to derive an equation for the minimizer

Theorem 2.7. *If $f \in \mathcal{F}$ is such that $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ minimizes (2.16), then f is a solution of the following equation:*

$$\Pi\Big(\Big(\frac{(1-\Omega^2)}{2}|z|^2 + Na|f|^2 e^{-|z|^2} - \mu\Big) f\Big) = 0, \tag{2.29}$$

where μ is the Lagrange multiplier coming from the L^2 constraint.

The equation for the minimizer allows us to derive that this minimizer cannot be a polynomial:

Theorem 2.8. *If $f \in \mathcal{F}$ is such that $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ minimizes (2.16), then f has an infinite number of zeroes.*

We expect that the minimizer is close to $\alpha\eta e^{-\Omega|z|^2/2}$, where η is the periodic function on the lattice (2.23) and α the inverted parabola (2.26) with Na replaced by Nab. Of course, this test function is not in our space of holomorphic functions, but $\Pi(\alpha\eta)e^{-\Omega|z|^2/2}$ is. We would like to get more information on the minimizer using this kind of tool.

2.3 Flow around an obstacle

In Chapter 7, we address the problem of a superfluid flow around an obstacle. It can be formulated as follows: understand the properties of the solutions of

$$\Delta\psi - 2ic\partial_x\psi + (\rho_0 - |\psi|^2)\psi = 0, \tag{2.30}$$

for $\mathbf{x} = (x, y)$ in $\omega = \mathbf{R}^2 \setminus \overline{B_1}$, where B_1 is the obstacle, and $\psi = 0$ on ∂B_1. Here c is the velocity of the flow at infinity and ρ_0 some fixed number. If the flow is dissipationless, that is, for small c, we expect the existence of a stationary stable solution of this equation, while if c is increased, the flow becomes time-dependent and vortices are nucleated. We expect the nonexistence of stationary solutions to hold. Our main result consists in a rigorous proof of the existence of stationary solutions of (2.30) for small c, such that ψ does not have vortices:

Theorem 2.9. *There exists* $c_0 > 0$ *such that for all* $c \in (0, c_0)$, *problem (2.30) has a vortex-free solution* ψ_c, *that is,* $|\psi_c| > 0$ *in* ω.

We also deal with a case closer to the experiments, in three dimensions, where ρ_0 is a function of position, and prove a similar theorem for small velocity. This setting and our numerical simulations provide many open questions that we try to formulate.

3

Two-Dimensional Model for a Rotating Condensate

In this chapter, we want to study the shape of the minimizers $u = u_\varepsilon \in H_0^1(\mathcal{D}; \mathbf{C})$ of

$$E_\varepsilon(u) = \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega \mathbf{r}^\perp \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2} \Big(|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}) \Big)^2 \right\} dx\, dy, \quad (3.1)$$

where $\mathbf{r} = (x, y)$, $\mathbf{r}^\perp = (-y, x)$, $(iu, \nabla u) = i(\bar{u}\nabla u - u\nabla\bar{u})/2$, ε is a small parameter, and Ω is the given rotational velocity. We assume that $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - r^2$, $\mathcal{D}$ is the disc of radius $R_0 = \sqrt{\rho_0}$ in $\mathbf{R}^2$ (so that $\rho_{\mathrm{TF}} = 0$ on $\partial\mathcal{D}$), and $\int_{\mathcal{D}} \rho_{\mathrm{TF}} = 1$, which prescribes the value of ρ_0. The issue is to determine the number and location of vortices according to the value of Ω.

As we have explained in the introduction, the energy formulation relies on two reductions: a two-dimensional reduction and a bounded-domain reduction. The two-dimensional reduction of the problem has been used in a number of physics papers [35, 45, 106]. There, the minimizer is computed either numerically or using a special ansatz (which corresponds mathematically to constructing an upper bound), and the critical velocities for the nucleation of vortices are determined. The minimization in a bounded domain $\mathcal{D}$ is not necessary to get the results, and a full two-dimensional analysis in $\mathbf{R}^2$ taking into account the mass constraint has been performed by Ignat and Millot [80, 81]. We follow their ideas for this simplified problem. For the ease of presentation, we restrict here to the model case $\rho_{\mathrm{TF}}(r) = \rho_0 - r^2$, but more general functions ρ_{TF} can be dealt with, as we will see in Chapter 4.

We want to prove that for small Ω, the minimizer has no vortices, and as Ω increases, determine the number and location of vortices. The analysis will be made in the framework of the book of Bethuel, Brezis, Helein [32], described for this setting in [12] and analyzed in Ignat–Millot [80, 81]: vortices are identified as balls where u is small and in which there is a degree. The method relies on an asymptotic expansion of the energy: each vortex located at p_i provides an energy contribution of order $\pi|\log\varepsilon|\rho_{\mathrm{TF}}(|p_i|)$, while the rotation term provides a negative counterpart: $-\Omega\rho_{\mathrm{TF}}^2(|p_i|)/2$. A vortex becomes energetically favorable when the sum of these two terms becomes negative; hence the critical velocity will be of order $|\log\varepsilon|$. The main tools that we are going to use have been developed by many authors in the context of Ginzburg–Landau vortices. Let us point out the contributions of Sandier–Serfaty [134, 135, 136, 143, 144], summarized in the book [138], but also of Lassoued–Mironescu [97] and André–Shafrir [22]. The main difference with previous work is that the potential term ρ_{TF} vanishes on the boundary of the domain, and we will not be able to say anything about vortices close to the boundary; hence we will have to restrict our analysis to $\mathcal{D}_\delta = \{\mathbf{r} \in \mathcal{D},\ \operatorname{dist}(|\mathbf{r}|^{\mathbf{2}}, \partial\mathcal{D}) > \delta\}$.

3.1 Main results

If the velocity is smaller than a critical velocity, we prove that there are no vortices in the system:

Theorem 3.1. *Let u_ε be a sequence of minimizers of E_ε in $H_0^1(\mathcal{D})$ and assume that Ω depends on ε in such a way that*

$$\lim_{\varepsilon\to 0} \frac{\Omega}{|\log \varepsilon|} = \omega_0. \tag{3.2}$$

Then $\omega_0^* = 2/\rho_0$ is a critical value in the sense that if $\omega_0 < \omega_0^*$, for any $\delta > 0$, if ε is smaller than some ε_δ, then u_ε does not vanish in $\mathcal{D}_\delta$. In addition, as ε tends to 0, $|u_\varepsilon|$ converges to $\sqrt{\rho_{\mathrm{TF}}}$ in $L^\infty_{\mathrm{loc}}(\mathcal{D})$, and

$$E_\varepsilon(u_\varepsilon) = \mathcal{E}(\varepsilon) + o(1), \tag{3.3}$$

where $\mathcal{E}(\varepsilon)$ does not depend on u_ε or Ω.

If ω_0 reaches the critical value ω_0^*, then the number of vortices in the system depends on the next term in the expansion of Ω:

Theorem 3.2. *We assume a specific asymptotic form for the rotation Ω:*

$$\Omega = \omega_0^* |\log \varepsilon| + \omega_1 \log|\log \varepsilon|. \tag{3.4}$$

Let u_ε be a sequence of minimizers of E_ε in $H_0^1(\mathcal{D})$:

(i) If $\omega_1 < 0$, then the conclusion of Theorem 3.1 holds.
(ii) If $\omega_1^n < \omega_1 < \omega_1^{n+1}$, with $\omega_1^n = 2(n-1)/\rho_0$, for any $\delta > 0$, if ε is smaller than some ε_δ, then u_ε has exactly n vortices p_i^ε of degree one in $\mathcal{D}_\delta$. Moreover,

$$|p_i^\varepsilon| < C/\sqrt{\Omega} \quad \textit{for any } i \textit{ and } |p_i^\varepsilon - p_j^\varepsilon| > C/\sqrt{\Omega},$$

where C is independent of ε. Let $\tilde{p}_i^\varepsilon = p_i^\varepsilon/\sqrt{\Omega}$. Then the configuration $\tilde{p}_i^\varepsilon$ tends to minimize the energy w defined in $\mathbf{R}^{2n}$ by

$$w(b_1, \ldots, b_n) = -\pi\rho_0 \sum_{i\neq j} \log|b_i - b_j| + \frac{\pi\rho_0}{2} \sum_i |b_i|^2. \tag{3.5}$$

We have the following asymptotic expansion for the energy:

$$E_\varepsilon(u_\varepsilon) = \mathcal{E}(\varepsilon) + \frac{\pi}{2} n\rho_0(n - 1 - \omega_1\rho_0)\log|\log\varepsilon| \\ + \min_{\mathbf{R}^{2n}} w + C_n + o(1), \tag{3.6}$$

where $\mathcal{E}(\varepsilon)$ does not depend on u_ε, and C_n is an explicit constant that depends only on n.

For $\omega_0 > \omega_0^*$, we do not perform the analysis here, but we expect a limiting free-boundary problem described in Open Problem 3.4 in Section 3.8, which gives rise to two regions: the central region has a dense distribution of vortices and the outer region has no vortices. The case of larger velocities, that is, of order c/ε^2, will be addressed in Chapter 5.

Our result does not include statements about vortices close to the boundary, since they lie in a region where their contribution to the energy is smaller than our precision of expansion: the energy contribution of vortices in $\mathcal{D} \setminus \mathcal{D}_\delta$ is of order δ, hence very small. A more precise expansion would only allow the elimination of vortices closer to the boundary, and different tools should be introduced in order to derive the nonexistence of vortices for sufficiently small velocities:

Open Problem 3.1 *There exists a constant C such that for $\Omega < C$, minimizers of E_ε do not have vortices in $\mathcal{D}$.*

One may hope to get this result by proving the uniqueness of the minimizer for small Ω and then using the rotational invariance. A first step is to get the nondegeneracy of the solution at $\Omega = 0$ and use an implicit function theorem to derive the uniqueness for Ω sufficiently small. The result should hold in a more general sense, that is for any velocity of order 1. For Ω larger, that is, of order $|\log\varepsilon|$, yet smaller than the first critical velocity, it is possible that the minimizer has vortices close to the boundary in the region of low density, arranged on a circle for instance. At the moment, neither numerical simulations nor analytical results give a hint.

3.1.1 Single-vortex solution and location of vortices

When the solution has a single vortex at the point p, the asymptotic expansion of the energy is simplified:

$$E_\varepsilon(u_\varepsilon) = \mathcal{E}(\varepsilon) + \pi|\log\varepsilon|\rho_{\mathrm{TF}}(p) - \pi\Omega\rho_{\mathrm{TF}}^2(p)/2 + O(1). \tag{3.7}$$

The location of the vortex is determined by the minimum of $E(p) = \rho_{\mathrm{TF}}(p) - \Omega\rho_{\mathrm{TF}}^2(p)/(2|\log\varepsilon|)$. For Ω small, the energy $E(p)$ is a decreasing function of the position p; hence the best situation is to have the vortex at the boundary, that is, no vortex at all. For Ω larger, the energy $E(p)$ has a local minimum when the vortex is at the origin but the global minimum is at the boundary. For Ω even larger, the global minimum is achieved for a vortex at the origin. From this, we see easily that there are two critical velocities: one corresponds to the single vortex at the origin being a local minimizer and the next one a global minimizer. This latter corresponds to our critical velocity ω_0^*, and the first vortex appears at the origin.

For a higher number of vortices, they are still very close to the origin (at distance of order $1/\sqrt{|\log \varepsilon|}$), and their precise location is determined by the minimization of w. This has been studied by Gueron–Shafrir [76] and their results are consistent with the experimental data such as those illustrated in Figure 1.3.

3.1.2 Ideas of the proof

The method relies on an asymptotic expansion of the energy: we have to construct an upper bound and a lower bound for the energy. The first step is inspired by an idea of Lassoued and Mironescu [97] and consists in removing from the energy the contribution due to the inhomogeneity of ρ_{TF}.

We first analyze the energy minimizers η_ε when $\Omega = 0$. Up to a multiplication of a complex number of modulus one, this minimizer is unique and real-valued, and $E_\varepsilon(\eta) = F_\varepsilon(\eta)$, where

$$F_\varepsilon(\eta) = \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla \eta|^2 + \frac{1}{4\varepsilon^2}(|\eta|^2 - \rho_{\mathrm{TF}}(r))^2 \right\} dx\, dy. \tag{3.8}$$

The minimizer η_ε of F_ε is (up to a complex multiplier of modulus one) the unique positive solution of

$$\Delta \eta_\varepsilon + \frac{1}{\varepsilon^2}\eta_\varepsilon(\rho_{\mathrm{TF}}(r) - \eta_\varepsilon^2) = 0 \quad \text{in } \mathcal{D}, \quad \eta_\varepsilon = 0 \quad \text{on } \partial\mathcal{D}. \tag{3.9}$$

Moreover, η_ε^2 converges to ρ_{TF} in $L^2(\mathcal{D})$ and uniformly on any compact set, but there is a boundary layer of size $\varepsilon^{2/3}$ due to the bad convergence of the gradient. Then, we define $v = u/\eta_\varepsilon$ and split the energy E_ε into the energy of the density profile η_ε and a reduced energy of the complex phase. Thus, we get our key identity:

$$E_\varepsilon(u) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_{\eta_\varepsilon}(v), \quad \text{where } \mathcal{E}_{\eta_\varepsilon}(v) = G_{\eta_\varepsilon}(v) + L_{\eta_\varepsilon}(v) \tag{3.10}$$

and

$$G_{\eta_\varepsilon}(v) = \int_{\mathcal{D}} \left\{ \frac{\eta_\varepsilon^2}{2}|\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2 - 1)^2 \right\} dx\, dy, \tag{3.11}$$

$$L_{\eta_\varepsilon}(v) = -\int_{\mathcal{D}} \eta_\varepsilon^2 \Omega \mathbf{r}^\perp \cdot (iv, \nabla v)\, dx\, dy. \tag{3.12}$$

The term G_{η_ε} is very similar to the energy studied by [32], with the addition of a weight. When η_ε is replaced by ρ_{TF}, we will use the following notations

$$\mathcal{E}_\varepsilon(v) = \mathcal{E}_{\rho_{\mathrm{TF}}}(v), \quad G_\varepsilon(v) = G_{\rho_{\mathrm{TF}}}(v), \quad L_\varepsilon(v) = L_{\rho_{\mathrm{TF}}}(v). \tag{3.13}$$

If the integration is not performed in $\mathcal{D}$ but in a smaller domain, it will be mentioned. We will study the vortex structure of u through the analysis of the map v and the energies G_ε and L_ε. Due to the degeneracy of the weight close to the boundary, difficulties arise in the region where ρ_{TF} is small. Hence our analysis will provide information only in the region $\mathcal{D}_\varepsilon = \{\rho_{\mathrm{TF}} > \nu|\log\varepsilon|^{-3/2}\}$ for some ν.

From (3.10), we expect to prove the following expansion of energy:

$$\begin{aligned} E_\varepsilon(u) = E_\varepsilon(\eta_\varepsilon) + \Big(\pi\rho_0 n|\log\varepsilon| - \frac{\pi\Omega}{2}n\rho_0^2\Big) \\ + \frac{\pi}{2}n\rho_0(n-1)\log\Omega + \min_{\mathbf{R}^{2n}} w + C + o(1), \end{aligned} \tag{3.14}$$

where n is the number of vortices. The second term in this expansion yields the value of the critical rotational velocity. Indeed, if it positive, it is better not to have vortices in the system, while as soon as it gets nonpositive, vortices become favorable. The third term in the expansion of the energy provides the next significant term in the expansion of Ω; thus it is natural to assume a special behaviour for the rotational velocity:

$$\Omega = \omega_0|\log\varepsilon| + \omega_1\log|\log\varepsilon|, \tag{3.15}$$

so that ω_1 will control the number of vortices.

Let us now explain how to derive (3.14). Using η_ε as a test function and (3.10), we immediately find that $\mathcal{E}_{\eta_\varepsilon}(v)$ is negative and $\mathcal{E}_{\eta_\varepsilon}(v,\mathcal{D}_\varepsilon)$ tends to 0 as ε tends to 0. The next step in the proof consists in deriving a first lower bound, inspired by [136], which allows us to characterize vortices through balls B_i centered at p_i carrying some degree d_i and some amount of energy:

$$G_\varepsilon(v, B_i) \geq \pi|d_i||\log\varepsilon|\rho_{\mathrm{TF}}(p_i). \tag{3.16}$$

This definition of vortex balls was introduced by Sandier [133] and Jerrard [84]. We find that as soon as Ω is bounded by $C|\log\varepsilon|$, there is a finite collection I_ε of such vortex balls in the system. Outside these balls, $|v|$ is close to 1, and we get an estimate of the rotational energy L_ε there, namely

$$L_\varepsilon(v, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_\varepsilon} B_i) \sim -\frac{\pi}{2}\Omega\sum_{i\in I_\varepsilon} d_i\rho_{\mathrm{TF}}^2(p_i). \tag{3.17}$$

The term ρ_{TF}^2 is specific to the harmonic potential: it comes from an integration by parts around the vortex balls, which requires a primitive of the function $r\rho_{\mathrm{TF}}(r)$. In the case of the harmonic potential, this primitive is proportional to ρ_{TF}^2.

From (3.16)–(3.17) and $\mathcal{E}_{\eta_\varepsilon}(v,\mathcal{D}_\varepsilon) = o(1)$, we find that

$$\sum_{i\in I_\varepsilon}\rho_{\mathrm{TF}}(p_i)\left(|\log\varepsilon||d_i| - \frac{\Omega\rho_0}{2}d_i\right) \leq o(1).$$

This provides the value of the critical velocity $\Omega_c = 2|\log\varepsilon|/\rho_0$, under which there are no vortices in $\mathcal{D}_\varepsilon$. We also obtain that as soon as ω_1 in (3.15) is bounded, the number of vortex balls with nonzero degree is uniformly bounded in ε and they are located close to the origin. We also get some improved energy bounds that will allow us to improve the vortex description, in particular

$$G_\varepsilon(v, \mathcal{D}_\varepsilon) \le C_{\omega_1}|\log\varepsilon|, \; G_\varepsilon(v, \mathcal{D}_\varepsilon \setminus \{|r| < 2|\log\varepsilon|^{-1/6}\})) \le C_{\omega_1}\log|\log\varepsilon|. \tag{3.18}$$

The next step consists in refining the upper and lower bounds to deduce a better expansion of the energy. This relies on a finer description of the vortex structure using the method of bad discs introduced by Bethuel, Brezis, and Helein [32]. These bad discs are smaller than the vortex balls defined above. Their number is uniformly bounded and they lie close to the origin. The main ingredients are the energy estimates (3.18) and a local version of the Pohozaev identity. The clustering method of [35] provides a new family of modified bad discs $B(x_j^\varepsilon, \rho)$, $j \in \tilde{\mathcal{F}}_\varepsilon$, with $\rho \sim \varepsilon^\alpha$ for some $\alpha \in (0, 1)$, and $\mathrm{Card}\tilde{\mathcal{F}}_\varepsilon$ bounded. We identify vortices with the points x_j^ε: outside the discs, $|v| > 1/2$, and v has nonzero degree D_i on the circles $\partial B(x_j^\varepsilon, \rho)$.

Following methods introduced in [32], we are able to improve the lower bound of the energy and evaluate the energy carried by each vortex,

$$G_\varepsilon(v, B(x_j^\varepsilon)) \ge \pi\rho_{\mathrm{TF}}(x_j^\varepsilon)|D_j|\log\frac{\rho}{\varepsilon} + O(1), \tag{3.19}$$

and the energy away from the vortices, which contains the interaction term

$$G_\varepsilon(v, B_R \setminus \cup_{j\in\tilde{\mathcal{F}}_\varepsilon} B(x_j^\varepsilon)) \ge \pi \sum_{j\in\tilde{\mathcal{F}}_\varepsilon} \rho_{\mathrm{TF}}(x_j^\varepsilon) D_j^2 |\log\rho| + W_{R,\varepsilon}((x_j^\varepsilon, D_j)) + O(1), \tag{3.20}$$

where $W_{R,\varepsilon}$ is a renormalized energy taking into account the interaction between the points. The radius R is fixed at this stage of the proof; hence the error term is a constant depending on R. It is only in the last step that we will let R tend to $\sqrt{\rho_0}$.

The next step consists in constructing an upper bound with d vortices located at a distance of order $1/\sqrt{|\log\varepsilon|}$ from the origin, on a lattice minimizing w, the expected limit of $W_{R,\varepsilon}$ as R tends to $\sqrt{\rho_0}$. The construction is inspired by [22].

Finally, the combination of the upper and lower bounds yields that for each j, $D_j = 1$, $\mathrm{Card}\tilde{\mathcal{F}}_\varepsilon = d$, and that the vortices are uniformly distributed at distance $1/\sqrt{|\log\varepsilon|}$ from the origin. We rescale the location of vortices and study the limit as ε tends to 0 of

$$E_\varepsilon(u) - E_\varepsilon(\eta_\varepsilon) - \left(\pi\rho_0 n|\log\varepsilon| - \frac{\pi\Omega}{2}n\rho_0^2\right) - \frac{\pi}{2}n\rho_0(n-1)\log\Omega.$$

Our lower bound is a function of R that we let tend to $\sqrt{\rho_0}$ to obtain the expansion (3.14).

This chapter is organized as follows: Section 3.2 contains the study of η_ε and the proof of (3.10). Then, in Section 3.3, we define the structure of vortex balls, and obtain the first lower bounds (3.16)–(3.17) and the estimate (3.18). Section 3.4 is devoted to the construction of refined estimates of the vortex structure based on the analysis of bad discs [32] and (3.18). The lower bound proved in Section 3.5 provides the interaction term $W_{R,\varepsilon}$. Then, in Section 3.6, we construct a test function and obtain the upper bound. Finally, in Section 3.7, we combine our upper and lower bounds to conclude with all the required estimates. Section 3.8 contains some open questions.

3.2 Preliminaries

This section is devoted to the study of minimizers of the reduced energy F_ε defined by (3.8), the existence of minimizers of E_ε, and the proof of the key identity (3.10), which is a first step towards the energy expansion.

3.2.1 Determining the density profile

Firstly, we study the minimizers η_ε of F_ε, which provide the shape of the density profile.

Proposition 3.3. *Problem (3.9) admits a unique positive solution η_ε, which is the unique minimizer of F_ε in $H^1_0(\mathcal{D})$ up to a complex multiplier of modulus one. In addition,*

(i) $\eta_\varepsilon \in C^\infty(\mathcal{D})$ is radial;
(ii) $0 < \eta_\varepsilon(r) \le \max_{\mathcal{D}} \rho_{\mathrm{TF}}$, and $|\nabla \eta_\varepsilon| \le C/\varepsilon$;
(iii) $F_\varepsilon(\eta_\varepsilon) \le C|\log \varepsilon|$ and $F_\varepsilon(\eta_\varepsilon, .)$ is bounded in $L^\infty_{\mathrm{loc}}(\mathcal{D})$.
(iv) There exists a constant C independent of ε such that

$$|\eta_\varepsilon(r) - \sqrt{\rho_{\mathrm{TF}}(r)}| \le C\varepsilon^{1/3}\sqrt{\rho_{\mathrm{TF}}(r)}\ \forall \mathbf{r} \in \mathcal{D} \text{ with } \operatorname{dist}(\mathbf{r}, \partial\mathcal{D}) \ge \varepsilon^{1/3}, \quad (3.21)$$

where $C > 0$ is a constant independent of ε.

The proof of (iii) relies on the expected size of the boundary layer, namely $\varepsilon^{2/3}$. There, η_ε should be close to the solution of (1.16).

The assertion (iv) implies that $|\eta_\varepsilon^2(r) - \rho_{\mathrm{TF}}(r)|$ is small with respect to $\rho_{\mathrm{TF}}(r)$ itself at a small distance from the boundary of $\mathcal{D}$.

Remark 3.4. We also have $\eta_\varepsilon^2 \to \rho_{\mathrm{TF}}$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathcal{D})$, $\|\eta_\varepsilon - \sqrt{\rho}_{\mathrm{TF}}\|_{C^1(K)} \le C_K\varepsilon^2$, for any compact subset K of $\mathcal{D}$.

Proof of Proposition 3.3: The existence of a positive minimizer of F_ε in $H^1_0(\mathcal{D})$ is standard. Since $F_\varepsilon(|\eta|) \le F_\varepsilon(\eta)$, with equality if and only if $\eta = |\eta|e^{i\alpha}$, the minimizer is a real positive function up to multiplication by a number of modulus 1 and satisfies the Euler–Lagrange equation. The uniqueness comes from [42]. Let us recall the proof briefly. If ξ and η are two solutions, then $w = \xi/\eta$ satisfies an equation that we multiply by $w - 1$ and integrate over $\mathcal{D}$ to obtain that $w \equiv 1$.

(i): by the uniqueness, η must be radial.

(ii): The maximum principle yields that $\eta > 0$ in $\mathcal{D}$ and $\eta < \max_{\mathcal{D}} \rho_{\mathrm{TF}}$. The estimate on the gradient follows from the equation and the Gagliardo–Nirenberg inequality as in [33].

(iii): Since η_ε is the minimizer of F_ε, we just need to construct a test function for which we have a bound on the energy. We define $\xi(r) = \gamma(\rho_{\mathrm{TF}}(r))$, where

$$\gamma(s) = \begin{cases} \sqrt{s}, & \text{if } s > \varepsilon^{2/3}, \\ \frac{s}{\varepsilon^{1/3}}, & \text{if } s < \varepsilon^{2/3}. \end{cases}$$

Using the coarea formula, we obtain

$$\int_{\mathcal{D}} |\nabla \xi|^2 = \int_{r_0}^{R_0} \gamma'(\rho_{\mathrm{TF}}(r))^2 |\nabla \rho_{\mathrm{TF}}|^2 \, dr \leq C \int_0^{\bar{a}} \gamma'(s)^2 \, ds \leq C |\log \varepsilon|.$$

For the other term,

$$\int_{\mathcal{D}} (\rho_{\mathrm{TF}} - \gamma(\rho_{\mathrm{TF}})^2)^2 \, dr \leq \int_0^{\varepsilon^{2/3}} (s - \gamma(s)^2)^2 \, ds \leq C\varepsilon^2.$$

Hence, the energy of this test function is bounded by $|\log \varepsilon|$.

In order to get the energy bound on compact sets, we follow [97] and fix δ such that $K_\delta = \{x \in \mathcal{D}, \mathrm{dist}(x, \partial K) < \delta\}$ is included in $\mathcal{D}$. We have in particular $F_\varepsilon(\eta_\varepsilon, K_\delta \setminus K) \leq C|\log \varepsilon|$. Hence there exists a compact K' containing K such that $F_\varepsilon(\eta_\varepsilon, \partial K') \leq C'|\log \varepsilon|$. We can assume that $K' = (r_0, r_1)$ and let $K'_\varepsilon = (r_0 + \varepsilon, r_1 - \varepsilon)$. We consider the following test function:

$$v_\varepsilon = \begin{cases} \eta_\varepsilon, & \text{in } \mathcal{D} \setminus \overline{K}', \\ \sqrt{\rho_{\mathrm{TF}}}, & \text{in } K'_\varepsilon, \\ \sqrt{t\eta_\varepsilon^2(r_0) + (1-t)\rho_{\mathrm{TF}}(r_0+\varepsilon)}, & \text{if } r \in (r_0, r_0+\varepsilon) \text{ and} \\ & \rho_{\mathrm{TF}}(r) = (1-t)\rho_{\mathrm{TF}}(r_0+\varepsilon) + t\rho_{\mathrm{TF}}(r_0), \\ \sqrt{t\eta_\varepsilon^2(r_1) + (1-t)\rho_{\mathrm{TF}}(r_1-\varepsilon)}, & \text{in } r \in (r_1-\varepsilon, r_1) \text{ and} \\ & \rho_{\mathrm{TF}}(r) = (1-t)\rho_{\mathrm{TF}}(r_1-\varepsilon) + \rho_{\mathrm{TF}}(r_1). \end{cases}$$

Using that $F_\varepsilon(\eta_\varepsilon) \leq F_\varepsilon(v_\varepsilon)$ and that v_ε and η_ε are equal in $\mathcal{D} \setminus \overline{K}'$, we obtain $F_\varepsilon(\eta_\varepsilon, K') \leq F_\varepsilon(v_\varepsilon, K')$. A computation of $F_\varepsilon(v_\varepsilon, K')$ together with the hypothesis $F_\varepsilon(\eta_\varepsilon, \partial K') \leq C'|\log \varepsilon|$ gives the result.

(iv): (3.21) is obtained by restricting to small balls $B_\delta(x_0)$ on which we construct subsolutions and supersolutions in the spirit of [22]. We refer to [3] or [80, 81] for more details. □

3.2.2 Existence of a minimizer of E_ε

The energy E_ε is not positive but we obtain a bound from below thanks to an estimate of the momentum term by the energy F_ε:

Lemma 3.5. *For any $u \in H_0^1(\mathcal{D})$, any $\sigma > 0$, we have*

$$\left|\Omega \int_{\mathcal{D}} \mathbf{r}^{\perp} \cdot (iu, \nabla u)\right| \leq \sigma F_{\varepsilon}(u) + \frac{C}{\sigma}\Omega^2 + \frac{C}{\sigma^3}\varepsilon^2\Omega^4, \tag{3.22}$$

where C depends only on ρ_{TF}, hence on $\mathcal{D}$, and F_{ε} is given by (3.8).

Proof: We have

$$\begin{aligned}\left|\Omega \int_{\mathcal{D}} \mathbf{r}^{\perp} \cdot (iu, \nabla u)\right| &\leq \frac{\sigma}{2}\int_{\mathcal{D}} |\nabla u|^2 + \frac{\Omega^2}{2\sigma}\int_{\mathcal{D}} |x|^2|u|^2 \\ &\leq \frac{\sigma}{2}\int_{\mathcal{D}} |\nabla u|^2 + \frac{\Omega^2}{2\sigma}\int_{\mathcal{D}} |x|^2\rho_{\mathrm{TF}} + \frac{\Omega^2}{2\sigma}\int_{\mathcal{D}} |x|^2(|u|^2 - \rho_{\mathrm{TF}}) \\ &\leq \sigma F_{\varepsilon}(u) + \frac{\Omega^2}{2\sigma}\int_{\mathcal{D}} |x|^2\rho_{\mathrm{TF}} + \frac{\varepsilon^2\Omega^4}{4\sigma^3}\int_{\mathcal{D}} |x|^4. \qquad \square\end{aligned}$$

This identity allows us to get the existence of minimizers of E_{ε} and some simple properties:

Proposition 3.6. *Assume that $\Omega < \Lambda|\log \varepsilon|$. Then there exists a minimizer u_{ε} of E_{ε} in $H_0^1(\mathcal{D})$. Moreover, u_{ε} satisfies*

$$\Delta u_{\varepsilon} - 2i\Omega \mathbf{r}^{\perp} \cdot \nabla u_{\varepsilon} + \frac{1}{\varepsilon^2}(\rho_{\mathrm{TF}}(x) - |u_{\varepsilon}|^2)u_{\varepsilon} = 0. \tag{3.23}$$

We have, for ε sufficiently small:

(a) $E_{\varepsilon}(u_{\varepsilon}) \leq C_{\Lambda}|\log \varepsilon|$, $F_{\varepsilon}(u_{\varepsilon}) \leq C_{\Lambda}|\log \varepsilon|^2$.
(b) $0 < |u_{\varepsilon}(x)| \leq \sqrt{\rho_{\mathrm{TF}}(x)}$.
(c) $\|\nabla u_{\varepsilon}\|_{L^{\infty}(K)} \leq C_{\omega_0,K}\, \varepsilon^{-1}$ for any compact subset K.

Proof: Lemma 3.5 with $\sigma = 1/2$ implies that

$$F_{\varepsilon}(u) \leq 2E_{\varepsilon}(u) + C_{\Lambda}|\log \varepsilon|^2. \tag{3.24}$$

The coercivity of F_{ε} implies the existence of a minimizer for E_{ε}. Using η_{ε} as a test function, we find that if u_{ε} is a minimizer, $E_{\varepsilon}(u_{\varepsilon}) \leq E_{\varepsilon}(\eta_{\varepsilon})$; hence from (3.24) and Proposition 3.3, (a) holds. For (b), we write the equation for $l_{\varepsilon} = |u_{\varepsilon}|^2$ and use the maximum principle. Finally, (c) comes from the Gagliardo Nirenberg inequality. $\square$

3.2.3 Splitting the energy

We use the idea of [97] to decouple the energy E_{ε} of any function u into that of the profile η_{ε} and the energy due to the vortex contribution and rotation.

Lemma 3.7. *Let $u \in H_0^1(\mathcal{D})$. Then $v = u/\eta_{\varepsilon}$ is well defined and (3.10) holds.*

Proof: Note that v is well defined in $\mathcal{D}$, since $\eta_\varepsilon > 0$. Since η_ε satisfies (3.9), we multiply it by $\eta_\varepsilon(1-|v|^2)$ and integrate:

$$\int_{\mathcal{D}} (|v|^2-1)\Big(-\frac{1}{2}\Delta\eta_\varepsilon^2 - \frac{1}{\varepsilon^2}\eta_\varepsilon^2(\rho_{\mathrm{TF}}-\eta_\varepsilon^2) + |\nabla\eta_\varepsilon|^2\Big) = 0. \tag{3.25}$$

Moreover,

$$\begin{aligned} E_\varepsilon(v\eta_\varepsilon) = {}& E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_{\eta_\varepsilon}(v) + \frac{1}{2}|\nabla\eta_\varepsilon|^2(|v|^2-1) + \eta_\varepsilon\nabla\eta_\varepsilon\cdot\nabla|v|^2 \\ & + \frac{1}{4\varepsilon^2}(\rho_{\mathrm{TF}}-\eta_\varepsilon^2|v|^2)^2 - \frac{1}{4\varepsilon^2}(\rho_{\mathrm{TF}}-\eta_\varepsilon^2)^2 - \frac{1}{4\varepsilon^2}\eta_\varepsilon^4(1-|v|^2)^2. \end{aligned}$$

This together with (3.25) yields (3.10). □

3.3 Bounded number of vortices

Our aim is to find a lower bound for the energy, which provides a definition and location of the vortex balls. This requires a bound on

$$\int_{\mathcal{D}} \frac{1}{2}|\nabla v|^2 + \frac{1}{4\varepsilon^2}(1-|v|^2)^2.$$

Because of the degeneracy of the weight, we cannot estimate this integral in the whole domain, and have to restrict to a subdomain at some distance of $\partial\mathcal{D}$. Let

$$\delta = \delta_\varepsilon = \nu|\log\varepsilon|^{-3/2} \text{ for some } \nu \in (1,2) \tag{3.26}$$

and

$$\mathcal{D}_\varepsilon := \{x \in \mathcal{D} : \ \mathrm{dist}(|x|^2, \partial\mathcal{D}) > \delta_\varepsilon\}, \qquad \mathcal{N}_\varepsilon := \mathcal{D}\setminus\mathcal{D}_\varepsilon.$$

Theorem 3.1 will follow from the first part of this proposition:

Proposition 3.8. *If u_ε is a minimizer of E_ε and Ω is of the type (3.15):*

(i) If either $\omega_0 < \omega_0^* = 2/\rho_0$ or $\omega_0 = \omega_0^*$ and $\omega_1 < -K_0$, then for any $\delta > 0$, for ε sufficiently small, u_ε does not vanish in $\mathcal{D}_\varepsilon$. Moreover, $|v_\varepsilon|$ tends to 1 locally uniformly in $\mathcal{D}_\varepsilon$ and $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = o(1)$.
(ii) If $\omega_0 = \omega_0^*$ and ω_1 is bounded, then there is a finite collection $\{B_i = B(p_i, s_i)\}_{i\in I_\varepsilon}$ of disjoint balls such that for $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$,

$$\left\{x \in \mathcal{D}_\varepsilon \ : \ |v_\varepsilon| < 1 - |\log\varepsilon|^{-5}\right\} \subset \bigcup_{i\in I_\varepsilon} B_i\ ; \tag{3.27}$$

$$\sum_{i\in I_\varepsilon} s_i < |\log\varepsilon|^{-10}; \tag{3.28}$$

Let

$$I_0 = \left\{ i \in I_\varepsilon \ : \ d_i \geq 0 \text{ and } |p_i| < |\log \varepsilon|^{-1/6} \right\},$$
$$I_* = \left\{ i \in I_\varepsilon \ : \ d_i \geq 0 \text{ and } |p_i| \geq |\log \varepsilon|^{-1/6} \right\},$$
$$I_- = \left\{ i \in I_\varepsilon \ : \ d_i < 0 \right\}.$$

Then there exists a constant C_{ω_1} depending only on ω_1 such that for ε sufficiently small,

$$N_0 := \sum_{i \in I_0} |d_i| \leq C_{\omega_1}, \tag{3.29}$$

and if $\mathcal{B}_\varepsilon = \left\{ x \in \mathcal{D} \ : \ \rho_{\mathrm{TF}}(x) \geq |\log \varepsilon|^{-1/2} \right\}$, then

$$\sum_{i \in I_* \cup I_-,\ p_i \in \mathcal{B}_\varepsilon} |d_i| = 0, \tag{3.30}$$

$$G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq C_{\omega_1} |\log \varepsilon|, \tag{3.31}$$

$$G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \{|x| < 2|\log \varepsilon|^{-1/6}\}) \leq C_{\omega_1} \log|\log \varepsilon|. \tag{3.32}$$

This section is devoted to the proof of this proposition.

3.3.1 First energy bound

Recall the definitions (3.10) and (3.13).

Proposition 3.9. *For ε small, we have*

$$|\mathcal{E}_\varepsilon(v, \mathcal{D}_\varepsilon) - \mathcal{E}_{\eta_\varepsilon}(v, \mathcal{D}_\varepsilon)| \to 0. \tag{3.33}$$

Moreover,

$$\mathcal{E}_\varepsilon(v, \mathcal{D}_\varepsilon) \leq c/|\log \varepsilon|, \tag{3.34}$$

$$\int_{\mathcal{D}_\varepsilon} \frac{\rho_{\mathrm{TF}}}{2} |\nabla v|^2 + \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2} (|v|^2 - 1)^2 \leq C |\log \varepsilon|^2,$$

$$\left| \Omega \int_{\mathcal{D}_\varepsilon} \rho_{\mathrm{TF}} \mathbf{r}^\perp \cdot (iv, \nabla v) \right| \leq C |\log \varepsilon|^2. \tag{3.35}$$

Proof: By (3.10) and the fact that $E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(\eta_\varepsilon)$, we deduce that $\mathcal{E}_{\eta_\varepsilon}(v) \leq 0$. Let us estimate the energy in $\mathcal{N}_\varepsilon$. We have

$$\begin{aligned}
\Omega \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 \mathbf{r}^\perp \cdot (iv, \nabla v) &\leq \frac{1}{2} \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 |\nabla v|^2 + \frac{1}{2} \Omega^2 \int_{\mathcal{N}_\varepsilon} |v|^2 \eta_\varepsilon^2 |x|^2 \\
&\leq \frac{1}{2} \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 |\nabla v|^2 + C\Omega^2 \int_{\mathcal{N}_\varepsilon} \left[\eta_\varepsilon^2 (|v|^2 - 1) + (\eta_\varepsilon^2 - \rho_{\mathrm{TF}}(r)) + \rho_{\mathrm{TF}}(r) \right] \\
&\leq \frac{1}{2} \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 |\nabla v|^2 + \int_{\mathcal{N}_\varepsilon} \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 + C\Omega^4 |\mathcal{N}_\varepsilon| \varepsilon^2 \\
&\quad + C\Omega^2 \left(\varepsilon \sqrt{F_\varepsilon(\eta_\varepsilon) |\mathcal{N}_\varepsilon|} + \delta_\varepsilon |\mathcal{N}_\varepsilon| \right) \\
&\leq \frac{1}{2} \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 |\nabla v|^2 + \int_{\mathcal{N}_\varepsilon} \frac{\eta_\varepsilon^4}{4\varepsilon^2} (|v|^2 - 1)^2 + C\Omega^2 \delta_\varepsilon^2.
\end{aligned} \tag{3.36}$$

In particular,

$$\mathcal{E}_{\eta_\varepsilon}(v;\mathcal{N}_\varepsilon)\geq -C/|\log\varepsilon|,$$

and consequently,

$$\mathcal{E}_{\eta_\varepsilon}(v;\mathcal{D}_\varepsilon)\leq C/|\log\varepsilon| \tag{3.37}$$

for any minimizer. Using the estimates on $(\eta_\varepsilon^2-\rho_{\mathrm{TF}})$ in $\mathcal{D}_\varepsilon$ from Proposition 3.3 (iv), we conclude that (3.33) and (3.34) hold.

Note that by similar steps to (3.36) above, we obtain

$$\Omega\int_{\mathcal{D}_\varepsilon}\eta_\varepsilon^2\mathbf{r}^\perp\cdot(iv,\nabla v)\leq\frac{1}{4}\int_{\mathcal{D}_\varepsilon}\eta_\varepsilon^2|\nabla v|^2+\frac{1}{8\varepsilon^2}\int_{\mathcal{D}_\varepsilon}\eta_\varepsilon^4(|v|^2-1)^2+C\Omega^2, \tag{3.38}$$

and hence from (3.37) we obtain

$$F_\varepsilon(v,\mathcal{D}_\varepsilon)=\int_{\mathcal{D}_\varepsilon}\frac{1}{2}\eta_\varepsilon^2|\nabla v|^2+\frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2\leq C\Omega^2=O(|\log\varepsilon|^2), \tag{3.39}$$

with C independent of ε. Moreover, the bounds (3.38), (3.39) also hold with ρ_{TF} replacing η_ε^2. □

3.3.2 Vortex balls

With the help of (3.35), we may isolate the vortex balls in in $\mathcal{D}_\varepsilon$ using the method of Sandier [133] and Sandier and Serfaty [135]:

Proposition 3.10. *For any $\Lambda>0$ there exist positive constants ε_0, C_0 such that for any $\varepsilon<\varepsilon_0$, $\Omega\leq\Lambda|\log\varepsilon|$, and any v satisfying (3.35), there exists a finite collection $\{B_i=B(p_i,s_i)\}_{i\in I_\varepsilon}$ of disjoint balls such that (3.27)–(3.28) hold as well as*

$$\int_{B_i}\frac{\rho_{\mathrm{TF}}}{2}|(\nabla-i\Omega\mathbf{r}^\perp)v|^2\geq\pi\rho_{\mathrm{TF}}(p_i)|d_i|\left(|\log\varepsilon|-C_0\log|\log\varepsilon|\right), \tag{3.40}$$

where

$$d_i=\deg_{\partial B_i}\left(\frac{v}{|v|}\right)\quad \textit{for all } i.$$

This implies that the set where u vanishes is contained in the balls B_i, and we have a size estimate and an energy estimate. The reason to restrict to $\mathcal{D}_\varepsilon$ is the need of an estimate of ρ_{TF} from below.

Proof: We sketch the proof, since the details are minor modifications of the analogous results in [135]. First, we complete the square in the gradient term and using (3.35) obtain

$$\int_{\mathcal{D}_\varepsilon}\left(\frac{\rho_{\mathrm{TF}}}{2}|(\nabla-i\Omega\mathbf{r}^\perp)v|^2+\frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(|v|^2-1)^2\right)$$
$$\leq\mathcal{E}_\varepsilon(v,\mathcal{D}_\varepsilon)+\frac{1}{2}\Omega^2\int_{\mathcal{D}_\varepsilon}\rho_{\mathrm{TF}}|x|^2|v|^2+o(1)\leq C|\log\varepsilon|^2.$$

Since $\rho_{\mathrm{TF}}(x) \geq \delta_\varepsilon$ for $x \in \mathcal{D}_\varepsilon$, using $\rho = |v|$, we have

$$\delta_\varepsilon^2 \int_{\mathcal{D}_\varepsilon} \left(\frac{1}{2}|\nabla \rho|^2 + \frac{1}{4\varepsilon^2}(\rho^2 - 1)^2 \right) \leq C|\log \varepsilon|^2,$$

and hence

$$\int_{\mathcal{D}_\varepsilon} \left(\frac{1}{2}|\nabla \rho|^2 + \frac{1}{4\varepsilon^2}(\rho^2 - 1)^2 \right) \leq C|\log \varepsilon|^5.$$

Let $\mathcal{D}_{\delta,t} := \{x \in \mathcal{D}_\varepsilon : \ \rho < 1 - t\}$, and $\gamma_t = \partial \mathcal{D}_{\delta,t}$. Using the coarea formula as in [135], there exists $t_0 \in (0, |\log \varepsilon|^{-5})$ and a finite set of balls $B_1, \dots, B_k$ with radii $s_1, \dots, s_k$ that cover γ_{t_0}, satisfying $\sum_i s_i \leq C\varepsilon|\log \varepsilon|^8$. In $\mathcal{D}_\varepsilon \setminus \mathcal{D}_{\delta,t_0}$ we may write $v = \rho e^{i\phi}$ for a (possibly multivalued) H^1_{loc} function $\phi(x)$.

Then we let the balls grow continuously, using the process described in [133], [135], to obtain as a final lower bound

$$\int_{B_i \setminus \mathcal{D}_{\delta,t_0}} \frac{\rho_{\mathrm{TF}}}{2}|\nabla \phi - \Omega \mathbf{x}|^2 \geq \pi \left(\min_{B_i} \rho_{\mathrm{TF}} \right) |d_i| \left(|\log \varepsilon| - \overline{C}_0 \log|\log \varepsilon| \right),$$

with constant $\overline{C}_0$ independent of ε. Note that the minimum of $\rho_{\mathrm{TF}}(x)$ over B_i is non-increasing as the radii increase and as balls are merged. We end the process when the sum of the radii of the balls equals $|\log \varepsilon|^{-10}$. By continuity of $\rho_{\mathrm{TF}}(x)$ we may then replace the minimum of ρ_{TF} on each ball by the value at its center p_i, making an error that is small compared to $\rho_{\mathrm{TF}}(p_i)$ itself. This error can then be absorbed into the coefficient of $\log|\log \varepsilon|$.

Finally,

$$\begin{aligned}
\int_{B_i} \frac{\rho_{\mathrm{TF}}}{2}|(\nabla - i\Omega \mathbf{r}^\perp)v|^2 &\geq \int_{B_i \setminus \mathcal{D}_{\delta,t_0}} \frac{\rho_{\mathrm{TF}}}{2}(1 + \rho^2 - 1)|\nabla \phi - \Omega \mathbf{r}^\perp|^2 \\
&\geq (1 - C|\log \varepsilon|^{-4}) \int_{B_i \setminus \mathcal{D}_{\varepsilon,t_0}} \frac{\rho_{\mathrm{TF}}}{2}|\nabla \phi - \Omega \mathbf{r}^\perp|^2 \\
&\geq (1 - o(1)) \left(\pi \rho_{\mathrm{TF}}(p_i)|d_i|(|\log \varepsilon| - \overline{C}_0 \log|\log \varepsilon|) \right) \\
&\geq \pi \rho_{\mathrm{TF}}(p_i)|d_i|(|\log \varepsilon| - C_0 \log|\log \varepsilon|),
\end{aligned}$$

for some constant C_0 independent of ε.

Note that by slightly modifying our choice of δ_ε we may be sure that no vortex ball intersects the boundary $\partial \mathcal{D}_\varepsilon$. If this is not the case, by (3.28) we may find a constant $k_\varepsilon \in [1, 2)$ such that replacing $\delta' = k_\varepsilon \delta_\varepsilon$ prevents vortex balls from intersecting the boundary. ◻

3.3.3 The rotation term

Lemma 3.11. *Let $\xi_\varepsilon(r) = (\rho_{\mathrm{TF}}^2(r) - \delta_\varepsilon^2)/4$. Note that $\xi_\varepsilon = 0$ on $\partial \mathcal{D}_\varepsilon$. We have*

$$\Omega \int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \rho_{\mathrm{TF}}(\mathbf{r})\, \mathbf{r}^\perp \cdot (iv, \nabla v) = \Omega \sum_i 2\pi d_i\, \xi_\varepsilon(|p_i|) + o(1). \tag{3.41}$$

Proof: In $\mathcal{D}_\varepsilon \setminus \cup B_i$, using Proposition 3.10, we may define $w = \frac{v}{|v|}$. Then $(iv, \nabla v) = |v|^2(iw, \nabla w)$, and $|\nabla v| \geq |v||\nabla w| \geq \frac{1}{2}|\nabla w|$ in $\mathcal{D}_\varepsilon \setminus \cup B_i$. Using the basic energy estimate (3.35), and the fact that $\rho_{\mathrm{TF}} \geq \delta_\varepsilon$ in $\mathcal{D}_\varepsilon$, we obtain

$$\begin{aligned}
&\left| \int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \rho_{\mathrm{TF}} \mathbf{r}^\perp \cdot (iv, \nabla v) - \int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \rho_{\mathrm{TF}} \mathbf{r}^\perp \cdot (iw, \nabla w) \right| \\
&\qquad = \left| \int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \rho_{\mathrm{TF}} (|v|^2 - 1) \mathbf{r}^\perp \cdot (iw, \nabla w) \right| \\
&\qquad \leq C \left(\int_{\mathcal{D}_\varepsilon} \rho_{\mathrm{TF}}^2 (|v|^2 - 1)^2 \right)^{1/2} \|\nabla w\|_2 \\
&\qquad \leq C\varepsilon |\log \varepsilon| \left(\frac{1}{\min_{\mathcal{D}_\varepsilon} \rho_{\mathrm{TF}}} \int_{\mathcal{D}_\varepsilon} \rho_{\mathrm{TF}} |\nabla v|^2 \right)^{1/2} \\
&\qquad \leq C \frac{\varepsilon |\log \varepsilon|^2}{\sqrt{\delta_\varepsilon}}.
\end{aligned} \tag{3.42}$$

Using the definition of ξ_ε, we have

$$\rho_{\mathrm{TF}}(\mathbf{r})\mathbf{r}^\perp = -\nabla^\perp \xi_\varepsilon(r).$$

Since $|w| = 1$, $(iw, \nabla w)$ is locally a gradient and is irrotational. Applying Stokes' theorem, we obtain

$$\begin{aligned}
\int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \rho_{\mathrm{TF}} \mathbf{r}^\perp \cdot (iw, \nabla w) &= \int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \nabla \xi_\varepsilon \cdot (iw, \nabla^\perp w) \\
&= -\int_{\partial \mathcal{D}_\varepsilon} \xi_\varepsilon (iw, \partial_\tau w) + \sum_i \int_{\partial B_i} \xi_\varepsilon(|x|)\,(iw, \partial_\tau w) \\
&= \sum_i \int_{\partial B_i} \xi_\varepsilon(|x|)\,(iw, \partial_\tau w).
\end{aligned}$$

The last equality uses that $\xi_\varepsilon = 0$ on $\partial \mathcal{D}_\varepsilon$.

To conclude, we need to approximate $\xi_\varepsilon(|x|)$ by $\xi_\varepsilon(|p_i|)$. This follows step by step from Lemma II.3 of [134] (see also [80]). Note that $|\nabla \xi_\varepsilon|$ is uniformly bounded independently of ε, and $\|\nabla w\|_2$ is bounded in terms of the energy using the same trick as in (3.42) above. We claim that for each vortex ball B_i of radius s_i,

$$\left| \Omega \int_{\partial B_i} (\xi_\varepsilon(|x|) - \xi_\delta(|p_i|))\ (iw, \partial_\tau w) \right| \leq \frac{|\log \varepsilon|^3}{\delta_\varepsilon} s_i\ ,$$

This allows us to conclude (3.41). $\square$

3.3.4 A lower bound expansion

Proof of Proposition 3.8: Note that because of (3.28),

$$\left|L_\varepsilon(v_\varepsilon, \cup_{i\in I_\varepsilon} B_i)\right| \le C|\log\varepsilon|^2 \sum_{i\in I_\varepsilon} r_i \le C|\log\varepsilon|^{-8}. \tag{3.43}$$

Putting (3.34), (3.40), and (3.41) together, we obtain the lower bound

$$\begin{aligned} C|\log\varepsilon|^{-1} &\ge \mathcal{E}_\varepsilon(v, \mathcal{D}_\varepsilon) \\ &\ge \pi \sum \rho_{\mathrm{TF}}(p_i)|d_i|\,(|\log\varepsilon| - C_0\log|\log\varepsilon|) - 2\pi\Omega \sum d_i \xi_\varepsilon(|p_i|) \\ &\quad + \frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup B_i} \rho_{\mathrm{TF}}\,|\nabla v|^2 + \int_{\mathcal{D}_\varepsilon} \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1-|v|^2)^2 + o(1). \end{aligned} \tag{3.44}$$

In particular, this implies

$$\begin{aligned} &\pi \sum_{i,\ d_i>0} \rho_{\mathrm{TF}}(p_i)|d_i| \Big(|\log\varepsilon|(1 - \frac{\omega_0\rho_0}{2} + \frac{\omega_0}{2}|p_i|^2) \\ &\quad - \log|\log\varepsilon|(C_0 + \frac{\omega_1}{2}\rho_{\mathrm{TF}}(|p_i|)) \Big) \\ &\quad + \pi \sum_{i,\ d_i<0} \rho_{\mathrm{TF}}(p_i)|d_i| \left(|\log\varepsilon| + \frac{\Omega}{2}\frac{\rho_{\mathrm{TF}}^2(|p_i|) - \delta^2}{\rho_{\mathrm{TF}}(|p_i|)} \right) \le C|\log\varepsilon|^{-1}. \end{aligned} \tag{3.45}$$

1st case: If $\omega_0 < \omega_0^* = 2/\rho_0$, then (3.45) implies that

$$\sum_i \rho_{\mathrm{TF}}(p_i)|d_i| \le C|\log\varepsilon|^{-2}. \tag{3.46}$$

Since in $\mathcal{D}_\varepsilon$, we can bound ρ_{TF} from below by $\delta = \nu|\log\varepsilon|^{-3/2}$, we find that $\sum_i |d_i| \le C|\log\varepsilon|^{-1/2}$. But $\sum_i |d_i|$ is an integer, so that it must be exactly zero for ε sufficiently small and there are no vortices in $\mathcal{D}_\varepsilon$. One can use that $\int \rho_{\mathrm{TF}}^2(1-|v|^2)^2/\varepsilon^2$ tends to zero to deduce that $|v|$ tends to 1 locally uniformly in $\mathcal{D}_\varepsilon$ and $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = o(1)$.

2nd case: If $\omega_0 = \omega_0^*$ and $\omega_1 < -K_0$, then (3.45) implies

$$\begin{aligned} &\pi \sum_{i,\ d_i>0} \rho_{\mathrm{TF}}(p_i)|d_i| \Big(|\log\varepsilon|\frac{\omega_0}{2}|p_i|^2 - (C_0 + \frac{\omega_1}{2}\rho_{\mathrm{TF}}(|p_i|))\log|\log\varepsilon| \Big) \\ &\quad + \pi \sum_{i,\ d_i<0} \rho_{\mathrm{TF}}(p_i)|d_i||\log\varepsilon| \\ &\quad + \frac{1}{2}\int_{\mathcal{D}_\varepsilon\setminus\cup B_i} \rho_{\mathrm{TF}}\,|\nabla v|^2 + \int_{\mathcal{D}_\varepsilon} \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1-|v|^2)^2 \le O(|\log\varepsilon|^{-1}). \end{aligned} \tag{3.47}$$

If $\omega_1 < -2C_0/\rho_0$, similar arguments as in the previous case yield that there are no vortices.

3rd case: $\omega_0 = \omega_0^*$. If ω_1 is too large, we are going to prove a bound on the number of vortices. We let

$$\tilde{I}_* = \{i \in I_* \ : \ p_i \in \mathcal{B}_\varepsilon\} \ , \quad N_* = \sum_{i \in \tilde{I}_*} |d_i|,$$

and

$$\tilde{I}_- = \{i \in I_- \ : \ p_i \in \mathcal{B}_\varepsilon\} \ , \quad N_- = \sum_{i \in \tilde{I}_-} |d_i|.$$

Since $\rho_{\mathrm{TF}}(p_i) \geq |\log \varepsilon|^{-1/2}$ for any $i \in \tilde{I}_* \cup \tilde{I}_-$, we obtain from (3.47),

$$\begin{aligned} \int_{\mathcal{D}_\varepsilon \setminus \cup_{i \in I_\varepsilon} B_i} & \rho_{\mathrm{TF}}(x) |\nabla v_\varepsilon|^2 + N_* |\log \varepsilon|^{1/6} + N_- |\log \varepsilon|^{1/2} \\ & \leq C(1+\omega_1) N_0 \log|\log \varepsilon| + \mathcal{O}(|\log \varepsilon|^{-1}), \end{aligned} \tag{3.48}$$

which implies in particular

$$\max\{N_*, N_-\} \leq C N_0 \frac{\log|\log \varepsilon|}{|\log \varepsilon|^{1/2}} \tag{3.49}$$

for ε sufficiently small. We now show that N_0 is uniformly bounded in ε. Consider the sets

$$\mathcal{I}_\varepsilon = \left[|\log \varepsilon|^{-1/6}, \frac{\sqrt{\rho_0}}{2} \right] \quad \text{and} \quad \mathcal{J}_\varepsilon = \left\{ r \in \mathcal{I}_\varepsilon \ : \ \partial B_r \cap (\cup_{i \in I_\varepsilon} \overline{B}_i) = \emptyset \right\}.$$

By Proposition 3.10, $\mathcal{J}_\varepsilon$ is a finite union of intervals satisfying $|\mathcal{I}_\varepsilon \setminus \mathcal{J}_\varepsilon| \leq |\log \varepsilon|^{-10}$. For each $r \in \mathcal{J}_\varepsilon$, $|v| \geq 1 - |\log \varepsilon|^{-5}$, and hence we may define

$$D(r) := \deg\left(\frac{v}{|v|}, \partial B_r(0)\right).$$

Moreover,

$$|D(r)| = \left| \sum_{|p_i| < r} d_i \right| \geq N_0 - N_- = N_0(1 - o(1)).$$

Writing $v = |v| e^{i\phi}$ (for $|x| = r \in \mathcal{J}_\varepsilon$), we estimate the remaining term in the energy as follows, using that in $\mathcal{J}_\varepsilon$, $|v| \geq 1 - |\log \varepsilon|^{-5}$:

$$\begin{aligned} \int_{\mathcal{D}_\varepsilon \setminus \cup B_i} \frac{\rho_{\mathrm{TF}}}{2} |\nabla v|^2 & \geq \int_{\mathcal{J}_\varepsilon} \int_0^{2\pi} \frac{\rho_{\mathrm{TF}}(r)}{2} |v|^2 |\nabla \phi|^2 \, r \, d\theta \, dr \\ & \geq \int_{\mathcal{J}_\varepsilon} \int_0^{2\pi} \frac{\rho_{\mathrm{TF}}(r)}{2} |\nabla \phi|^2 \, r \, d\theta \, dr (1 + o(1)) \end{aligned}$$

$$
\begin{aligned}
&\geq \pi \int_{\mathcal{J}_\varepsilon} \frac{\rho_{\mathrm{TF}}(r)}{r} \, (D(r))^2 \, (1+o(1)) \\
&\geq C \, N_0^2 \int_{\mathcal{J}_\varepsilon} \frac{dr}{r} \\
&\geq C \, N_0^2 \log|\log \varepsilon| (1+o(1)).
\end{aligned} \tag{3.50}
$$

Returning to (3.47), we obtain

$$
C_1 N_0^2 - C_2 N_0 \leq C \frac{|\log \varepsilon|^{-1}}{\log|\log \varepsilon|},
$$

with constants C_1, C_2 independent of ε. We conclude that N_0 is bounded. Thus, we infer from (3.49) that for ε small, N_- and N_* are zero; hence (3.30) holds.

Energy estimates

From Lemma 3.11 and (3.30), we deduce that for ε small,

$$
\begin{aligned}
L_\varepsilon\big(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i \in I_\varepsilon} B_i\big) \geq &- \frac{\pi \rho_0 \Omega}{2} \sum_{i \in I_0} \rho_{\mathrm{TF}}(p_i) |d_i| \\
&- \frac{\pi \Omega}{2} |\log \varepsilon|^{-1/2} \sum_{i \in I_* \setminus \tilde{I}_*} \rho_{\mathrm{TF}}(p_i) |d_i| + o(|\log \varepsilon|^{-5}) \\
\geq &-\pi \sum_{i \in I_0} \rho_{\mathrm{TF}}(p_i) |d_i| \Big(|\log \varepsilon| + \frac{\rho_0 \omega_1}{2} \log|\log \varepsilon| \Big) \\
&- \frac{2\pi}{\rho_0} \sum_{i \in I_*} \rho_{\mathrm{TF}}(p_i) |d_i| |\log \varepsilon|^{1/2} + o(|\log \varepsilon|^{-5}).
\end{aligned}
$$

We now use this estimate in (3.44) together with (3.43) to derive that

$$
\sum_{i \in I_*} \rho_{\mathrm{TF}}(p_i) |d_i| |\log \varepsilon| \leq C N_0 \log|\log \varepsilon|,
$$

and hence with (3.29), we conclude that $\sum_{i \in I_*} \rho_{\mathrm{TF}}(p_i) |d_i| |\log \varepsilon|^{1/2} = o(1)$. According to (3.43), this implies

$$
\begin{aligned}
L_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &= L_\varepsilon \left(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i \in I_\varepsilon} B_i \right) + o(1) \\
&\geq -\pi \sum_{i \in I_0} \rho_{\mathrm{TF}}(p_i) |d_i| \Big(|\log \varepsilon| + \frac{\rho_0 \omega_1}{2} \log|\log \varepsilon| \Big) + o(1).
\end{aligned}
$$

Since $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + L_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq \mathcal{O}(|\log \varepsilon|^{-1})$, we have

$$
\begin{aligned}
G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) &\leq \pi \sum_{i \in I_0} \rho_{\mathrm{TF}}(p_i) |d_i| \Big(|\log \varepsilon| + \frac{\rho_0 \omega_1}{2} \log|\log \varepsilon| \Big) + o(1) \qquad (3.51) \\
&\leq C_{\omega_1} N_0 |\log \varepsilon| \leq C_{\omega_1} |\log \varepsilon|.
\end{aligned}
$$

Let $\mathcal{A}_\varepsilon = \mathcal{D}_\varepsilon \setminus B_{2|\log\varepsilon|^{-1/6}}$. Matching (3.40) with (3.51), we finally obtain

$$\begin{aligned} G_\varepsilon(v_\varepsilon, \mathcal{A}_\varepsilon) &\le G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_0} B_i) \\ &\le \pi\Big(\frac{\rho_0\omega_1}{2} + K_0\Big) \sum_{i\in I_0} \rho_{\mathrm{TF}}(p_i)|d_i|\log|\log\varepsilon| + o(1) \\ &\le C_{\omega_1} N_0 \log|\log\varepsilon| \le C_{\omega_1}\log|\log\varepsilon|. \qquad \square \end{aligned}$$

3.4 Refined structure of vortices

Now that we know that for each ε there is a finite number of vortex balls, we want to locate them better and prove that they are close to the origin. The analysis here follows the ideas in [33] and [35], adapted to this setting by [81]. We prove that there is a finite number of bad discs (bounded independently of ε). The main difficulty is due to the presence in the energy of the weight function ρ_{TF}, which vanishes on $\partial\mathcal{D}$ and does not allow one to extend the structure up to the boundary. This section will be devoted to the proof of the following result:

Proposition 3.12. *Assume that u_ε is a minimizer of E_ε, $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, and Ω is given by (3.15).*
(1) *For any $R \in \left(\frac{\sqrt{\rho_0}}{2}, \sqrt{\rho_0}\right)$ there exists $\varepsilon_R > 0$ such that for any $\varepsilon < \varepsilon_R$,*

$$|v_\varepsilon| \ge \frac{1}{2} \quad \text{in } B_R \setminus B_{\frac{\sqrt{\rho_0}}{2}}.$$

(2) *There exist some constants $N \in \mathbf{N}$, $\lambda_0 > 0$, and $\varepsilon_0 > 0$ (that depend only on ω_1) such that for any $\varepsilon < \varepsilon_0$, there exists a finite collection of points $\{x_j^\varepsilon\}_{j\in\mathcal{F}_\varepsilon} \subset B_{\frac{\sqrt{\rho_0}}{4}}$ such that $\mathrm{Card}(\mathcal{F}_\varepsilon) \le N$ and*

$$|v_\varepsilon| \ge \frac{1}{2} \quad \text{in } \overline{B}_{\frac{\sqrt{\rho_0}}{2}} \setminus \Big(\cup_{j\in\mathcal{F}_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon)\Big).$$

Remark 3.13. The statement of Proposition 3.12 also holds if the radius $\frac{\sqrt{\rho_0}}{2}$ is replaced by an arbitrary $r \in (0, R)$, but then the constants in Proposition 3.12 depend on r. For simplicity, we fix $r = \frac{\sqrt{\rho_0}}{2}$.

The following proposition is a summary of the properties of bad discs that we will use for the lower bound:

Proposition 3.14. *Let $0 < \beta < \mu < 1$ be given constants such that $\overline{\mu} := \mu^{N+1} > \beta$ and let $\{x_j^\varepsilon\}_{j\in J_\varepsilon}$ be the collection of points given by (2) in Proposition 3.12. There exists $0 < \varepsilon_1 < \varepsilon_0$ such that for any $\varepsilon < \varepsilon_1$, we can find $\tilde{\mathcal{F}}_\varepsilon \subset \mathcal{F}_\varepsilon$ and $\rho > 0$ satisfying*

(i) $\lambda_0 \varepsilon \le \varepsilon^{\mu} \le \rho \le \varepsilon^{\overline{\mu}} < \varepsilon^{\beta}$,
(ii) $|v_\varepsilon| \ge \frac{1}{2}$ *in* $\overline{B}_{\frac{\sqrt{\rho_0}}{2}} \setminus \cup_{j \in \tilde{\mathcal{F}}_\varepsilon} B(x_j^\varepsilon, \rho)$,
(iii) $|v_\varepsilon| \ge 1 - \frac{2}{|\log \varepsilon|^2}$ *on* $\partial B(x_j^\varepsilon, \rho)$ *for every* $j \in \tilde{\mathcal{F}}_\varepsilon$,
(iv) $\int_{\partial B(x_j^\varepsilon, \rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \le \frac{C(\beta,\mu)}{\rho}$ *for every* $j \in \tilde{\mathcal{F}}_\varepsilon$,
(v) $|x_i^\varepsilon - x_j^\varepsilon| \ge 8\rho$ *for every* $i, j \in \tilde{\mathcal{F}}_\varepsilon$ *with* $i \ne j$.

Moreover, for each $j \in \tilde{\mathcal{F}}_\varepsilon$*, we have*

$$D_j := \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_j^\varepsilon, \rho)\right) \ne 0 \quad \text{and} \quad |D_j| \le C \tag{3.52}$$

for a constant C *independent of* ε.

3.4.1 Some local estimates

Our first lemma relies on the Pohozaev identity and will play a similar role as Theorem III.2 in [33]. In our situation, we derive only local estimates as in [35]. In the sequel, R denotes some arbitrary radius in $[\frac{\sqrt{\rho_0}}{2}, \sqrt{\rho_0})$ and we will write $R' = \frac{R+\sqrt{\rho_0}}{2}$.

Lemma 3.15. *For any* $\frac{2}{3} < \alpha < 1$*, there exists a positive constant* $C_{R,\alpha}$ *such that*

$$\frac{1}{\varepsilon^2} \int_{B(x_0, \varepsilon^\alpha)} (1 - |v_\varepsilon|^2)^2 \le C_{R,\alpha} \quad \textit{for any } x_0 \in B_R.$$

Proof: We proceed in several steps.
Step 1. We claim that

$$F_\varepsilon(u_\varepsilon, B_{R'}) \le C_R |\log \varepsilon|. \tag{3.53}$$

Indeed, since $u_\varepsilon = \eta_\varepsilon v_\varepsilon$, and η_ε is bounded below away from the boundary, we get that

$$\int_{B_{R'}} |\nabla u_\varepsilon|^2 \le C_R \left(\min_{B_{R'}} \rho_{\mathrm{TF}}\right)^{-1} \int_{B_{R'}} \rho_{\mathrm{TF}}(x) |\nabla v_\varepsilon|^2 + C_R \int_{B_{R'}} |\nabla \eta_\varepsilon|^2 \le C_R |\log \varepsilon|.$$

We also have

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{B_{R'}} (\rho_{\mathrm{TF}}(x) - |u_\varepsilon|^2)^2 &\le \frac{C}{\varepsilon^2} \int_{B_{R'}} \Big[(\rho_{\mathrm{TF}}(x) - \eta_\varepsilon^2)^2 + \eta_\varepsilon^4 (1 - |v_\varepsilon|^2)^2 \Big] \\ &\le \frac{C}{\varepsilon^2} \int_{B_{R'}} (\rho_{\mathrm{TF}}(x) - \eta_\varepsilon^2)^2 + \frac{C_R}{\varepsilon^2} \int_{B_{R'}} \rho_{\mathrm{TF}}^4(x) (1 - |v_\varepsilon|^2)^2 \\ &\le C_R |\log \varepsilon| \end{aligned}$$

because of Proposition 3.3 (iii) and (3.31). Therefore (3.53) follows.

Step 2. We are going to show that one can find a constant $C_{R,\alpha} > 0$, independent of ε, such that for any $x_0 \in B_R$, there is some $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ satisfying

$$F_\varepsilon(u_\varepsilon, \partial B(x_0, r_0)) \leq \frac{C_{R,\alpha}}{r_0}.$$

By contradiction, assume that for all $M > 0$, there is $x_M \in B_R$ such that

$$F_\varepsilon(u_\varepsilon, \partial B(x_M, r)) \geq \frac{M}{r}, \quad \forall r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3}). \tag{3.54}$$

Without loss of generality, we may assume that $B(x_M, \varepsilon^{\alpha/2+1/3}) \subset B_{R'}$. Integrating (3.54) over $r \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$, we derive that

$$F_\varepsilon(u_\varepsilon, B_{R'}) \geq M \int_{\varepsilon^\alpha}^{\varepsilon^{\alpha/2+1/3}} \frac{dr}{r} = M(-\alpha/2 + 1/3)|\log\varepsilon|,$$

which contradicts Step 1 for M large enough.

Step 3. Fix $x_0 \in B_R$ and let $r_0 \in (\varepsilon^\alpha, \varepsilon^{\alpha/2+1/3})$ be given by Step 2. As in Step 2, we may assume that $B(x_0, r_0) \subset B_{R'}$. We have

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2}(\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)u_\varepsilon + \frac{1}{\varepsilon^2}(\rho_{\mathrm{TF}}(x) - \rho_{\mathrm{TF}}(x_0))u_\varepsilon + 2i\Omega\mathbf{r}^\perp \cdot \nabla u_\varepsilon. \tag{3.55}$$

As in the proof of the Pohozaev identity, we multiply the various terms in (3.55) by $(x - x_0) \cdot \nabla\bar{u}_\varepsilon$, add the conjugate, and integrate by parts to get

$$\int_{B(x_0,r_0)} -\Delta u_\varepsilon \cdot [(x - x_0) \cdot \nabla u_\varepsilon] + c.c. = r_0 \int_{\partial B(x_0,r_0)} |\nabla u_\varepsilon|^2 - 2r_0 \int_{\partial B(x_0,r_0)} \left|\frac{\partial u_\varepsilon}{\partial\nu}\right|^2 \tag{3.56}$$

and

$$\begin{aligned}\frac{1}{\varepsilon^2} \int_{B(x_0,r_0)} &(\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)u_\varepsilon \cdot [(x - x_0) \cdot \nabla u_\varepsilon] + c.c.\\ &= \frac{1}{\varepsilon^2}\int_{B(x_0,r_0)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 - \frac{r_0}{2\varepsilon^2}\int_{\partial B(x_0,r_0)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2\end{aligned} \tag{3.57}$$

(where ν is the outer normal vector to $\partial B(x_0, r_0)$). From (3.55), (3.56), and (3.57) we derive that

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 \; &\leq C\Bigg(r_0 \int_{\partial B(x_0,r_0)} |\nabla u_\varepsilon|^2\\ &+ r_0 \int_{\partial B(x_0,r_0)} \varepsilon^{-2}(\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 + r_0\varepsilon^{-2}\int_{B(x_0,r_0)} |\rho_{\mathrm{TF}}(x) - \rho_{\mathrm{TF}}(x_0)||\nabla u_\varepsilon|\\ &+ \Omega r_0 \int_{B(x_0,r_0)} |\nabla u_\varepsilon|^2\Bigg).\end{aligned}$$

Then we estimate each integral term in the right hand side of the previous inequality. According to (3.53), we have

$$\int_{\partial B(x_0,r_0)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 \le C \int_{\partial B(x_0,r_0)} (\rho_{\mathrm{TF}}(x) - |u_\varepsilon|^2)^2 + C_R \varepsilon^{\frac{3}{2}\alpha+1},$$

$$\Omega r_0 \int_{B(x_0,r_0)} |\nabla u_\varepsilon|^2 \le \Omega r_0 F_\varepsilon(u_\varepsilon, B_{R'}) \le C_R\, \varepsilon^{\alpha/2+1/3} |\log \varepsilon|^2,$$

and

$$\begin{aligned} r_0 \varepsilon^{-2} \int_{B(x_0,r_0)} |\rho_{\mathrm{TF}}(x) - \rho_{\mathrm{TF}}(x_0)||u_\varepsilon||\nabla u_\varepsilon| &\le C_R\, r_0^2\, \varepsilon^{-2} \int_{B(x_0,r_0)} |\nabla u_\varepsilon| \\ &\le C_R\, r_0^3\, \varepsilon^{-2} [F_\varepsilon(u_\varepsilon, B_R)]^{1/2} \le C_R\, \varepsilon^{\frac{3}{2}\alpha-1} |\log \varepsilon|^{1/2} \end{aligned}$$

(here we use that $|\rho_{\mathrm{TF}}(x) - \rho_{\mathrm{TF}}(x_0)| \le C_R\, r_0$ for any $x, x_0 \in B_{R'}$). We finally get that

$$\frac{1}{\varepsilon^2} \int_{B(x_0,r_0)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 \le C_{R,\alpha} \big(1 + r_0 F_\varepsilon\, (u_\varepsilon, \partial B(x_0, r_0))\big)$$

for some constant $C_{R,\alpha}$ independent of ε. By Step 2, we conclude that

$$\frac{1}{\varepsilon^2} \int_{B(x_0,\varepsilon^\alpha)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 \le C_{R,\alpha}. \tag{3.58}$$

Using the estimates for η_ε, we have

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{B(x_0,\varepsilon^\alpha)} (1 - |v_\varepsilon|^2)^2 &\le \frac{C_R}{\varepsilon^2} \int_{B(x_0,\varepsilon^\alpha)} (\eta_\varepsilon^2 - |u_\varepsilon|^2)^2 \\ &\le \frac{C_R}{\varepsilon^2} \int_{B(x_0,\varepsilon^\alpha)} (\rho_{\mathrm{TF}}(x) - |u_\varepsilon|^2)^2 + o(1) \\ &\le \frac{C_R}{\varepsilon^2} \int_{B(x_0,\varepsilon^\alpha)} (\rho_{\mathrm{TF}}(x_0) - |u_\varepsilon|^2)^2 + o(1) \le C_{R,\alpha}, \end{aligned}$$

and we conclude with (3.58). □

3.4.2 Bad discs

The next result will allow us to define the notion of a bad disc as in [33].

Proposition 3.16. *There exist positive constants λ_R and μ_R such that if*

$$\frac{1}{\varepsilon^2} \int_{B_{R'} \cap B(x_0,2l)} (1 - |v_\varepsilon|^2)^2 \le \mu_R \quad \text{with } x_0 \in B_R\,,\ \frac{l}{\varepsilon} \ge \lambda_R\,,$$

then $|v_\varepsilon| \ge 1/2$ in $B_{R'} \cap B(x_0, l)$.

We refer to [32] for the proof, which relies on the fact that there exists a constant $C_R > 0$ independent of ε such that

$$|\nabla v_\varepsilon| \leq \frac{C_R}{\varepsilon} \quad \text{in } B_{R'}.$$

Definition 3.17. For $x \in B_R$, we say that $B(x, \lambda_R \varepsilon)$ is a **bad disc** if

$$\frac{1}{\varepsilon^2} \int_{B_{R'} \cap B(x, 2\lambda_R \varepsilon)} (1 - |v_\varepsilon|^2)^2 \geq \mu_R.$$

Now we can give a local version of Proposition 3.12. We will see that Lemma 3.15 plays a crucial role in the proof.

Proposition 3.18. *Let* $\frac{2}{3} < \alpha < 1$. *There exist positive constants* $N_{R,\alpha}$ *and* $\varepsilon_{R,\alpha}$ *such that for every* $\varepsilon < \varepsilon_{R,\alpha}$ *and* $x_0 \in B_R$ *one can find* $x_1, \dots, x_{N_\varepsilon} \in B(x_0, \varepsilon^\alpha)$ *with* $N_\varepsilon \leq N_{R,\alpha}$ *satisfying*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \textit{in} \quad B(x_0, \varepsilon^\alpha) \setminus \left(\cup_{k=1}^{N_\varepsilon} B(x_k, \lambda_R \varepsilon) \right).$$

Proof: We follow the ideas in [33], Chapter IV. First, choosing ε small enough, we may assume that $B(x_0, \varepsilon^\alpha) \subset B_{R'}$. Then, we use the covering lemma to obtain a family of discs $\left\{ B(x_i, \lambda_R \varepsilon) \right\}_{i \in \mathcal{F}}$ such that

$$\begin{cases} x_i \in B(x_0, \varepsilon^\alpha), \\ B\left(x_i, \frac{\lambda_R \varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_R \varepsilon}{4}\right) = \emptyset \quad \forall i \neq j, \\ B(x_0, \varepsilon^\alpha) \subset \bigcup_{i \in \mathcal{F}} B(x_i, \lambda_R \varepsilon). \end{cases} \tag{3.59}$$

We denote by $\mathcal{F}'$ the set of indices $i \in \mathcal{F}$ such that $B(x_i, \lambda_R \varepsilon)$ is a bad disc. We derive from Proposition 3.16 that

$$\mu_R \operatorname{Card}(\mathcal{F}') \leq \sum_{i \in \mathcal{F}} \frac{1}{\varepsilon^2} \int_{B_{R'} \cap B(x_i, 2\lambda_R \varepsilon)} (1 - |v_\varepsilon|^2)^2 \leq \frac{C}{\varepsilon^2} \int_{B(x_0, \varepsilon^\alpha)} (1 - |v_\varepsilon|^2)^2,$$

where C is some absolute constant. The conclusion now follows by Lemma 3.15. □

Remark 3.19. The proof of Proposition 3.18 implies that any collection of balls $\left\{ B(x_i, \lambda_R \varepsilon) \right\}_{i \in \mathcal{F}}$ satisfying (3.59) cannot contain more than $N_{R,\alpha}$ bad discs.

3.4.3 No degree-zero vortex

We need the following lemma to prove that vortices of degree zero do not occur. The main ingredients in the proof come from [35].

Lemma 3.20. *Let $D > 0$, $0 < \beta < 1$, and $\gamma > 1$ be given constants such that $\gamma\beta < 1$. Let $0 < \rho < \varepsilon^{\beta}$ be such that $\rho^{\gamma} > \lambda_R \varepsilon$. We assume that for $x_0 \in B_R$,*

(i) $\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 < \frac{D}{\rho}$,
(ii) $|v_\varepsilon| \geq \frac{1}{2}$ *on* $\partial B(x_0, \rho)$,
(iii) $\deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0,\rho)\right) = 0$.

Then we have

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B(x_0, \rho^{\gamma}).$$

Proof: The proof consists in constructing a comparison function as in [35], which allows us to obtain

$$\int_{B(x_0,\rho)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C_{\beta,R}. \tag{3.60}$$

We will not repeat it here. We deduce that

$$\int_{2\rho^{\gamma}}^{\rho} \left(\int_{\partial B(x_0,s)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2\right) ds \leq C_{\beta,R}.$$

Since $\int_{2\rho^{\gamma}}^{\rho} \frac{ds}{s|\log s|^{1/2}} \geq C_\gamma |\log\varepsilon|^{1/2}$, we derive that for small ε there exists $s_0 \in [2\rho^{\gamma}, \rho]$ such that

$$\int_{\partial B(x_0,s_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq \frac{C_{\beta,R}}{s_0|\log s_0|^{1/2}}\,.$$

Repeating the arguments used to prove (3.60), we find that

$$\int_{B(x_0,s_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq \frac{C_{\beta,R}}{|\log s_0|^{1/2}}.$$

In particular,

$$\frac{1}{\varepsilon^2}\int_{B(x_0,2\rho^{\gamma})} (1-|v_\varepsilon|^2)^2 = o(1),$$

and the conclusion follows by Proposition 3.16. □

We now obtain as in [35] Proposition IV.3 the following result, which provides an estimate of the energy contribution of any vortex.

Proposition 3.21. *Let $x_0 \in B_R$ and $\frac{2}{3} < \alpha < 1$. Assume that $|v_\varepsilon(x_0)| < \frac{1}{2}$. Then there exists a positive constant $C_{R,\alpha}$ (that depends only on R, α, and ω_1) such that*

$$\int_{B(x_0,\varepsilon^{\alpha})} |\nabla v_\varepsilon|^2 \geq C_{R,\alpha}|\log\varepsilon|.$$

Proof: Let $N_{R,\alpha}$ and $x_1, \ldots, x_{N_\varepsilon} \in B(x_0, \varepsilon^\alpha)$ be as in Proposition 3.18. Let

$$\delta_\alpha = \frac{\alpha^{1/2} - \alpha}{3(N_{R,\alpha} + 1)},$$

and for $k = 0, \ldots, 3N_{R,\alpha} + 2$ we consider

$$\alpha_k = \alpha^{1/2} - k\delta_\alpha\,,\ \mathcal{I}_k = [\varepsilon^{\alpha_k}, \varepsilon^{\alpha_{k+1}}],\ \text{ and } \mathcal{C}_k = B(x_0, \varepsilon^{\alpha_{k+1}}) \setminus B(x_0, \varepsilon^{\alpha_k}).$$

Then there is some $k_0 \in \{1, \ldots, 3N_{R,\alpha} + 1\}$ such that

$$\mathcal{C}_{k_0} \cap \left(\cup_{j=1}^{N_\varepsilon} B(x_j, \lambda_R \varepsilon) \right) = \varnothing. \tag{3.61}$$

Indeed, since $N_\varepsilon \le N_{R,\alpha}$ and $2\lambda_R\varepsilon < |\mathcal{I}_k|$ for small ε, the union of N_ε intervals of length $2\lambda_R\varepsilon$,

$$\cup_{j=1}^{N_\varepsilon} (|x_i - x_0| - \lambda_R\varepsilon, |x_i - x_0| + \lambda_R\varepsilon),$$

cannot intersect all the intervals $\mathcal{I}_k$ of disjoint interior, for $1 \le k \le 3N_{R,\alpha} + 1$. From (3.61) we deduce that

$$|v_\varepsilon(x)| \ge \frac{1}{2} \quad \forall x \in \mathcal{C}_{k_0}.$$

Therefore, for every $\rho \in \mathcal{I}_{k_0}$,

$$d_{k_0} = \deg\left(\frac{v_\varepsilon}{|v_\varepsilon|}, \partial B(x_0, \rho) \right)$$

is well defined and does not depend on ρ.

We claim that

$$d_{k_0} \neq 0. \tag{3.62}$$

By contradiction, we suppose that $d_{k_0} = 0$. According to (3.31),

$$\int_{B_{\frac{2\sqrt{\rho_0}+R}{3}}} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \le C_R |\log \varepsilon|.$$

Using the same argument as in Step 2 of the proof of Lemma 3.15, there is a constant $C_{R,\alpha}$ such that

$$\int_{\partial B(x_0,\rho_0)} |\nabla v_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \le \frac{C_{R,\alpha}}{\rho_0} \quad \text{for some } \rho_0 \in \mathcal{I}_{k_0}.$$

According to Lemma 3.20 (where $\beta = \alpha_{k_0+1}$ and $\gamma = \frac{\alpha_{k_0-1}}{\alpha_{k_0}}$), we should have $|v_\varepsilon(x_0)| \ge \frac{1}{2}$, which is a contradiction.

By (3.62), we obtain for every $\rho \in \mathcal{I}_{k_0}$,

$$1 \le |d_{k_0}| = \frac{1}{2\pi} \left| \int_{\partial B(x_0,\rho)} \frac{1}{|v_\varepsilon|^2} \big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau} \big) \right| \le C \int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|$$

(we use that $2 \geq |v_\varepsilon| \geq \frac{1}{2}$ in $\mathcal{C}_{k_0}$). The Cauchy–Schwarz inequality yields

$$\int_{\partial B(x_0,\rho)} |\nabla v_\varepsilon|^2 \geq \frac{C}{\rho} \quad \forall \rho \in \mathcal{I}_{k_0}$$

and the conclusion follows by integrating over $\mathcal{I}_{k_0}$. □

3.4.4 Proof of Proposition 3.12

Part (1) in Proposition 3.12 follows directly from Lemma 3.22 below.

Lemma 3.22. *There exists a constant* $\varepsilon_R > 0$ *such that for any* $0 < \varepsilon < \varepsilon_R$*,*

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } B_R \setminus B_{\frac{\sqrt{\rho_0}}{5}}.$$

Proof: First, we fix some $\alpha \in \left(\frac{2}{3}, 1\right)$. We proceed by contradiction. Suppose that there is some $x_0 \in B_R \setminus B_{\frac{\sqrt{\rho_0}}{5}}$ such that $|v_\varepsilon(x_0)| < 1/2$. Then for any ε sufficiently small, we have $B(x_0, \varepsilon^\alpha) \subset \mathcal{D}_\varepsilon \setminus \{|x| < 2|\log \varepsilon|^{-1/6}\}$ and therefore, by (3.32), we get that

$$\int_{B(x_0,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \leq C_R\, G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \{|x| < 2|\log \varepsilon|^{-1/6}\}) \leq C_R \log|\log \varepsilon|,$$

which contradicts Proposition 3.21 for ε small enough. □

Proof of (2) in Proposition 3.12. We fix some $\frac{2}{3} < \alpha < 1$. As in the proof of Proposition 3.18, we consider a finite family of points $\{x_j\}_{j \in \mathcal{F}}$ satisfying

$$\begin{aligned} & x_j \in B_{\frac{\sqrt{\rho_0}}{2}} \\ & B\left(x_i, \frac{\lambda_0 \varepsilon}{4}\right) \cap B\left(x_j, \frac{\lambda_0 \varepsilon}{4}\right) = \emptyset \quad \forall i \neq j\,, \\ & B_{\frac{\sqrt{\rho_0}}{2}} \subset \bigcup_{j \in \mathcal{J}} B\left(x_j, \lambda_0 \varepsilon\right), \end{aligned}$$

where $\lambda_0 := \lambda_{\frac{\sqrt{\rho_0}}{2}}$ (defined in Proposition 3.16 with $R = \frac{\sqrt{\rho_0}}{2}$) and we denote by $\mathcal{F}_\varepsilon$ the set of indices $j \in \mathcal{F}$ such that $B(x_j, \lambda_0 \varepsilon)$ contains at least one point y_j satisfying

$$|v_\varepsilon(y_j)| < \frac{1}{2}\,. \tag{3.63}$$

Obviously, $B(x_j, \lambda_0\varepsilon)$ is a bad disc when $j \in \mathcal{F}$. Applying Lemma 3.22 (with $R = \frac{3\sqrt{\rho_0}}{4}$), we infer that there exists ε_0 such that for any $0 < \varepsilon < \varepsilon_0$,

$$B(x_j, \lambda_0 \varepsilon) \subset B_{\frac{\sqrt{\rho_0}}{4}} \quad \text{for any } j \in \mathcal{F}_\varepsilon. \tag{3.64}$$

Then it remains to prove that Card$\mathcal{F}_\varepsilon$ is bounded independently of ε. Using Proposition 3.21 (with $R = \frac{\sqrt{a_0}}{2}$), we derive that for every $j \in \mathcal{F}_\varepsilon$ and any point y_j satisfying (3.63) in the ball $B(x_j, \lambda_0\varepsilon)$,

$$\int_{B(x_j,2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq \int_{B(y_j,\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha |\log \varepsilon| \tag{3.65}$$

for some positive constant C_α that depends only on α. For ε small enough, we define

$$W = \bigcup_{j \in J_\varepsilon} B(x_j, 2\varepsilon^\alpha) \subset B_{\frac{\sqrt{\rho_0}}{3}} .$$

We claim that there is a positive integer M_α independent of ε such that any $y \in W$ belongs to at most M_α balls in the collection $\{B(x_j, 2\varepsilon^\alpha)\}_{j \in J_\varepsilon}$. Indeed, for each $y \in W$, let

$$K_y = \left\{ j \in J_\varepsilon \ : \ y \in B(x_j, 2\varepsilon^\alpha) \right\}.$$

Then, for every $j \in K_y$,

$$x_j \in B(y, 2\varepsilon^\alpha) \subset B(y, \varepsilon^{\alpha'}) \subset B_{\frac{\sqrt{\rho_0}}{2}} \quad \text{with } \alpha' = \frac{1}{2}\left(\alpha + \frac{2}{3}\right). \tag{3.66}$$

The family of bad discs $\{B(x_j, \lambda_0\varepsilon)\}_{j \in K_y}$ is a subcover of $B(y, \varepsilon^{\alpha'})$ satisfying (3.59) and therefore, by Remark 3.19,

$$\mathrm{Card}(K_y) \leq M_\alpha$$

for $M_\alpha = N_{\sqrt{\rho_0}/2,\alpha'}$. From (3.65), we deduce that

$$\int_{B_{\frac{\sqrt{\rho_0}}{2}}} |\nabla v_\varepsilon|^2 \geq \int_W |\nabla v_\varepsilon|^2 \geq \frac{1}{M_\alpha} \sum_{j \in \mathcal{F}_\varepsilon} \int_{B(x_j,2\varepsilon^\alpha)} |\nabla v_\varepsilon|^2 \geq C_\alpha \mathrm{Card}(\mathcal{F}_\varepsilon) |\log \varepsilon|. \tag{3.67}$$

On the other hand, we know by (3.31),

$$\int_{B_{\frac{\sqrt{\rho_0}}{2}}} |\nabla v_\varepsilon|^2 \leq C \int_{B_{\frac{\sqrt{\rho_0}}{2}}} \rho_{\mathrm{TF}}(x) |\nabla v_\varepsilon|^2 \leq C |\log \varepsilon| \tag{3.68}$$

for a constant C independent of ε. Matching (3.67) and (3.68), we conclude that Card$\mathcal{F}_\varepsilon$ is uniformly bounded. ☐

Proof of Proposition 3.14: We proceed exactly as in [143]. By Proposition 3.12, we have for ε small enough,

$$\cup_{j \in J_\varepsilon} B(x_j^\varepsilon, \lambda_0\varepsilon) \subset B_{\frac{\sqrt{\rho_0}}{3}} .$$

From *(iii)* in Proposition 3.10, there exists a radius $r_\varepsilon \in (\frac{\sqrt{\rho_0}}{3}, \frac{\sqrt{\rho_0}}{2}]$ such that

$$\bar{B}_i \cap \partial B_{r_\varepsilon} = \emptyset \quad \text{for every } i \in I_\varepsilon. \tag{3.69}$$

Hence we have

$$|v_\varepsilon| \geq 1 - |\log \varepsilon|^{-5} \quad \text{on } \partial B_{r_\varepsilon}.$$

The existence of a subset $\tilde{\mathcal{F}}_\varepsilon \subset \mathcal{F}_\varepsilon$ satisfying *(i)–(v)* can now be proved identically as Proposition 3.2 in [143]. It remains to prove (3.52). From the proof of Proposition 3.12, we know (by construction) that each disc $B(x_k^\varepsilon, \lambda_0\varepsilon)$, $k \in \mathcal{F}_\varepsilon$, contains at least one point y_k such that $|v_\varepsilon(y_k)| < \frac{1}{2}$. Therefore each disc $B(x_j^\varepsilon, \rho)$, $j \in \tilde{\mathcal{F}}_\varepsilon$, contains at least one of the y_k's with $|x_j^\varepsilon - y_k| < \lambda_0\varepsilon$. Assume now that $D_j = 0$. By Lemma 3.20 with $\gamma = \mu^{-1/2}$, it would lead to $|v_\varepsilon| \geq \frac{1}{2}$ in $B(x_j^\varepsilon, \rho^\gamma)$ and then $|v_\varepsilon(y_k)| \geq \frac{1}{2}$ for ε small enough, which is impossible. We also obtain a bound on the degrees D_j:

$$|D_j| = \frac{1}{2\pi} \left| \int_{\partial B(x_j^\varepsilon, \rho)} \frac{1}{|v_\varepsilon|^2} \big(v_\varepsilon \wedge \frac{\partial v_\varepsilon}{\partial \tau} \big) \right| \leq C \|\nabla v_\varepsilon\|_{L^2(\partial B(x_j^\varepsilon, \rho))} \sqrt{\rho} \leq C$$

by *(iv)*. □

3.5 Lower bound

In this section, we obtain various lower energy estimates for v_ε in terms of the vortex structure defined in Proposition 3.14. We start by proving a lower bound on the gradient term away from the vortices, which brings out the interaction between vortices and eventually the lower bound for the whole energy. The method is based on the techniques developed in [33, 81, 143]. To avoid the difficulties due to the degeneracy of ρ_{TF} close to the boundary, the estimates will be proved in B_R for an arbitrary radius $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0}\,)$. To emphasize the possible dependence on R in the "error term," we will denote by $\mathcal{O}_R(1)$ (respectively $o_R(1)$) any quantity that remains uniformly bounded in ε for fixed R (respectively any quantity that tends to 0 as $\varepsilon \to 0$ for fixed R).

Proposition 3.23. *For any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0}\,)$, we have*

$$\begin{aligned} \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq{} & \pi \sum_{j=1}^{n_\varepsilon} D_j^2 \, \rho_{\mathrm{TF}}(x_j^\varepsilon) |\log \rho| + \pi \sum_{j=1}^{n_\varepsilon} |D_j| \, \rho_{\mathrm{TF}}(x_j^\varepsilon) \log \frac{\rho}{\varepsilon} \\ & - \frac{\pi \Omega}{2} \sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}^2(x_j^\varepsilon) \, D_j + W_{R,\varepsilon} + \mathcal{O}_R(1), \end{aligned} \tag{3.70}$$

where

$$\begin{aligned} W_{R,\varepsilon}\big((x_1^\varepsilon, D_1), \ldots, (x_{n_\varepsilon}^\varepsilon, D_{n_\varepsilon})\big) ={} & -\pi \sum_{i \neq j} D_i D_j \, \rho_{\mathrm{TF}}(x_j^\varepsilon) \log |x_i^\varepsilon - x_j^\varepsilon| \\ & - \pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{R,\varepsilon}(x_j^\varepsilon) \end{aligned}$$

and $\Psi_{R,\varepsilon}$ is the unique solution of

$$\begin{cases} div\,\left(\frac{1}{\rho_{\mathrm{TF}}}\nabla\Psi_{R,\varepsilon}\right) = -\sum_{j=1}^{n_\varepsilon} D_j\,\rho_{\mathrm{TF}}(x_j^\varepsilon)\,\nabla\left(\frac{1}{\rho_{\mathrm{TF}}}\right)\cdot\nabla\left(\log|x-x_j^\varepsilon|\right) \text{ in } B_R, \\ \Psi_{R,\varepsilon} = -\sum_{j=1}^{n_\varepsilon} D_j\,\rho_{\mathrm{TF}}(x_j^\varepsilon)\log|x-x_j^\varepsilon| \text{ on } \partial B_R. \end{cases} \tag{3.71}$$

Moreover, if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \neq j$ then the term $\mathcal{O}_R(1)$ in (3.73) is in fact $o_R(1)$. We also have

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} |D_j|\,\rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}^2(x_j^\varepsilon)\,D_j + \mathcal{O}(1). \tag{3.72}$$

Remark 3.24. We point out that the dependence on R in the interaction term $W_{R,\varepsilon}$ appears only in the function $\Psi_{R,\varepsilon}$.

Let us start with the estimate on the gradient term, which provides the interaction between vortices:

Proposition 3.25. *For any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0}\,)$, let $\Theta_\rho = B_R \setminus \cup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)$. Then*

$$\frac{1}{2}\int_{\Theta_\rho} \rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2 \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2\,\rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\rho| + W_{R,\varepsilon}\big((x_i^\varepsilon, D_i)\big) + \mathcal{O}_R(1). \tag{3.73}$$

Proof: We consider the solution Φ_ρ of the linear problem

$$\begin{cases} \mathrm{div}\big(\frac{1}{\rho_{\mathrm{TF}}}\nabla\Phi_\rho\big) = 0 & \text{in } \Theta_\rho, \\ \Phi_\rho = 0 & \text{on } \partial B_R, \\ \Phi_\rho = \text{const.} & \text{on } \partial B(x_j^\varepsilon, \rho), \\ \int_{\partial B(x_j^\varepsilon,\rho)} \frac{1}{\rho_{\mathrm{TF}}}\frac{\partial\Phi_\rho}{\partial\nu} = 2\pi D_j & \text{for } j = 1, \ldots, n_\varepsilon, \end{cases}$$

and $\Phi_{R,\varepsilon}$ the solution of

$$\begin{cases} \mathrm{div}\big(\frac{1}{\rho_{\mathrm{TF}}}\nabla\Phi_{R,\varepsilon}\big) = 2\pi\sum_{j=1}^{n_\varepsilon} D_j\,\delta_{x_j^\varepsilon} & \text{in } B_R, \\ \Phi_{R,\varepsilon} = 0 & \text{on } \partial B_R. \end{cases} \tag{3.74}$$

For $x \in \Theta_\rho$, we let $w_\varepsilon(x) = \frac{v_\varepsilon(x)}{|v_\varepsilon(x)|}$ and

$$S = \left(-w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_2} + \frac{1}{\rho_{\mathrm{TF}}}\frac{\partial\Phi_\rho}{\partial x_1}\,,\; w_\varepsilon \wedge \frac{\partial w_\varepsilon}{\partial x_1} + \frac{1}{\rho_{\mathrm{TF}}}\frac{\partial\Phi_\rho}{\partial x_2}\right).$$

We easily check that div $S = 0$ in Θ_ρ and $\int_{\partial B_{R_\varepsilon}} S\cdot\nu = \int_{\partial B(x_j^\varepsilon,\rho)} S\cdot\nu = 0$. By Lemma I.1 in [33], there exists $H \in C^1(\overline{\Theta}_\rho)$ such that $S = \nabla^\perp H$ and hence we can write the Hodge–de Rham type decomposition

$$w_\varepsilon \wedge \nabla w_\varepsilon = \frac{1}{\rho_{\mathrm{TF}}} \nabla^\perp \Phi_\rho + \nabla H.$$

Consequently,

$$\int_{\Theta_\rho} \rho_{\mathrm{TF}}(x)|\nabla w_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 + 2\int_{\Theta_\rho} \nabla^\perp \Phi_\rho \cdot \nabla H. \tag{3.75}$$

We observe that the last term is in fact equal to zero since Φ_ρ is constant on $\partial\Theta_\rho$. Since $|\nabla v_\varepsilon|^2 \geq |v_\varepsilon|^2 |\nabla w_\varepsilon|^2$ in Θ_ρ, we derive that

$$\int_{\Theta_\rho} \rho_{\mathrm{TF}}(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 + T_1$$

with

$$T_1 = \int_{\Theta_\rho} \big(|v_\varepsilon|^2 - 1\big) \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2.$$

It turns out that $T_1 = o_R(1)$ and therefore

$$\int_{\Theta_\rho} \rho_{\mathrm{TF}}(x)|\nabla v_\varepsilon|^2 \geq \int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 + o_R(1). \tag{3.76}$$

On the other hand, integrating by parts we obtain

$$\int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 = \int_{\partial\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} \frac{\partial \Phi_\rho}{\partial \nu} \Phi_\rho = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Phi_\rho(z_j)$$

for any point $z_j \in \partial B(x_j^\varepsilon, \rho)$. Since n_ε and each D_j remain uniformly bounded in ε by Proposition 3.14, we may rewrite this equality as

$$\int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 = -2\pi \sum_{j=1}^{n_\varepsilon} D_j \, \Phi_{R,\varepsilon}(z_j) + \mathcal{O}(\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)}). \tag{3.77}$$

Using an adaptation of Lemma I.4 in [33], we derive that

$$\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \leq \sum_{j=1}^{n_\varepsilon} \Big(\sup_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon, \rho)} \Phi_{R,\varepsilon} \Big). \tag{3.78}$$

To estimate the right-hand-side term in (3.78), we introduce for $x \in B_R$,

$$\Psi_{R,\varepsilon}(x) = \Phi_{R,\varepsilon}(x) - \sum_{j=1}^{n_\varepsilon} D_j \, \rho_{\mathrm{TF}}(x_j^\varepsilon) \log|x - x_j^\varepsilon|.$$

Since $\Phi_{R,\varepsilon}$ solves (3.74), we deduce that $\Psi_{R,\varepsilon}$ may be characterized as the solution of equation (3.71). By elliptic regularity, we infer that $\|\Psi_{R,\varepsilon}\|_{W^{2,p}(B_R)} \leq C_{R,p}$ for any $1 \le p < 2$ (here we have used that $\{x_j^\varepsilon\}_{j=1}^{n_\varepsilon} \subset B_{\frac{\sqrt{\rho_0}}{4}}$ by Proposition 3.12). In particular, $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R)$ and hence

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} - \inf_{\partial B(x_j^\varepsilon,\rho)} \Psi_{R,\varepsilon} \le C_R\sqrt{\rho} = o_R(1).$$

Since $|x_j^\varepsilon - x_i^\varepsilon| \ge 8\rho$, we derive from (3.52),

$$\sup_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i\, \rho_{\mathrm{TF}}(x_i^\varepsilon) \log|x - x_i^\varepsilon| \right) - \inf_{\partial B(x_j^\varepsilon,\rho)} \left(\sum_{i=1}^{n_\varepsilon} D_i\, \rho_{\mathrm{TF}}(x_i^\varepsilon) \log|x - x_i^\varepsilon| \right)$$
$$\le \rho \sum_{i=1}^{n_\varepsilon} \rho_{\mathrm{TF}}(x_i^\varepsilon) \sup_{\partial B(x_j^\varepsilon,\rho)} \frac{|D_i|}{|x - x_i^\varepsilon|} \le \mathcal{O}(1)$$

(respectively $\le o(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \ne j$). Returning to (3.78), we obtain that $\|\Phi_{R,\varepsilon} - \Phi_\rho\|_{L^\infty(\Theta_\rho)} \le \mathcal{O}_R(1)$ (respectively $\le o_R(1)$ if $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} \to 0$ as $\varepsilon \to 0$ for any $i \ne j$). Inserting this estimate in (3.77), we get that

$$\begin{aligned}
\int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 &= -2\pi \sum_{j=1}^{n_\varepsilon} D_j\, \Phi_{R,\varepsilon}(z_j) + \mathcal{O}_R(1) \\
&= -2\pi \sum_{j=1}^{n_\varepsilon} D_j\, \Psi_{R,\varepsilon}(z_j) - 2\pi \sum_{i \ne j} D_i D_j\, \rho_{\mathrm{TF}}(x_i^\varepsilon) \log|z_j - x_i^\varepsilon| \\
&\quad + 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2\, \rho_{\mathrm{TF}}(x_j^\varepsilon) |\log \rho| + \mathcal{O}_R(1)
\end{aligned} \tag{3.79}$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). Since $\Psi_{R,\varepsilon}$ is uniformly bounded with respect to ε in $C^{0,1/2}(B_R)$, we have $|\Psi_{R,\varepsilon}(z_j) - \Psi_{R,\varepsilon}(x_j^\varepsilon)| \le C_R\sqrt{\rho} = o_R(1)$. Moreover, using (3.52) and $|x_j^\varepsilon - x_i^\varepsilon| \ge 8\rho$, we derive that

$$\Big| \sum_{i \ne j} D_i D_j\, \rho_{\mathrm{TF}}(x_i^\varepsilon) (\log|z_j - x_i^\varepsilon| - \log|x_j^\varepsilon - x_i^\varepsilon|) \Big|$$
$$\le \sum_{i \ne j} |D_i|\,|D_j| \log\Big|1 + \frac{z_j - x_j^\varepsilon}{x_j^\varepsilon - x_i^\varepsilon}\Big| \le \sum_{i \ne j} |D_i|\,|D_j| \frac{\rho}{|x_j^\varepsilon - x_i^\varepsilon|} \le \mathcal{O}(1)$$

(respectively $\le o(1)$ as $\varepsilon \to 0$). Hence (3.79) yields

$$\begin{aligned}
\int_{\Theta_\rho} \frac{1}{\rho_{\mathrm{TF}}(x)} |\nabla \Phi_\rho|^2 &= -2\pi \sum_{j=1}^{n_\varepsilon} D_j\, \Psi_{R,\varepsilon}(x_j^\varepsilon) - 2\pi \sum_{i \ne j} D_i D_j\, \rho_{\mathrm{TF}}(x_i^\varepsilon) \log|x_j^\varepsilon - x_i^\varepsilon| \\
&\quad + 2\pi \sum_{j=1}^{n_\varepsilon} D_j^2\, \rho_{\mathrm{TF}}(x_j^\varepsilon) |\log \rho| + \mathcal{O}_R(1)
\end{aligned}$$

(respectively $+o_R(1)$ as $\varepsilon \to 0$). By combining this estimate with (3.76), we obtain the announced result. □

After estimating the contribution in the energy of each vortex, we may easily deduce the following lower bounds for $G_\varepsilon(v_\varepsilon)$:

Lemma 3.26. *For any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0})$, we have*

$$G_\varepsilon(v_\varepsilon, B_R) \geq \pi \sum_{j=1}^{n_\varepsilon} D_j^2\, \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log \rho| + \pi \sum_{j=1}^{n_\varepsilon} |D_j|\, \rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} + W_{R,\varepsilon} + \mathcal{O}_R(1) \tag{3.80}$$

and also

$$G_\varepsilon(v_\varepsilon, B_R) \geq \pi \sum_{j=1}^{n_\varepsilon} |D_j|\, \rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} + \mathcal{O}(1). \tag{3.81}$$

Let us point out that in the second estimate, the rest $\mathcal{O}(1)$ does not depend on R.

Proof: In view of Proposition 3.25 or the fact that G_ε is positive outside the balls, it is sufficient to show that

$$G_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) \geq \pi |D_j|\, \rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} + \mathcal{O}(1) \ \text{ for } j = 1, \dots, n_\varepsilon,$$

which is equivalent to

$$\frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2 \geq \pi |D_j| \log\frac{\rho}{\varepsilon} + \mathcal{O}(1) \ \text{ for } j = 1, \dots, n_\varepsilon \tag{3.82}$$

(we have used that $|\rho_{\mathrm{TF}}(x) - \rho_{\mathrm{TF}}(x_j^\varepsilon)| \leq C\rho$ for $x \in B(x_j^\varepsilon, \rho)$ and $G_\varepsilon(v_\varepsilon, B_R) \leq C|\log\varepsilon|$). Let

$$\hat{v}(y) = v_\varepsilon(\rho y + x_j^\varepsilon) \quad \text{for } y \in B(0,1) \ \text{ and } \quad \hat{\varepsilon} = \frac{\varepsilon}{\rho\sqrt{\rho_{\mathrm{TF}}(x_j^\varepsilon)}}.$$

We deduce from Proposition 3.14 that $|\hat{v}| \geq 1 - \frac{2}{|\log\,\varepsilon|}$ on $\partial B(0,1)$,

$$\int_{\partial B(0,1)} \frac{|\nabla \hat{v}|^2}{2} + \frac{1}{4\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2 = \int_{\partial B(x_j^\varepsilon,\rho)} \frac{|\nabla v_\varepsilon|^2}{2} + \frac{\rho_{\mathrm{TF}}(x_j^\varepsilon)}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 \leq C, \tag{3.83}$$

and

$$\frac{1}{2}\int_{B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2 = \frac{1}{2}\int_{B(x_j^\varepsilon,\rho)} |\nabla v_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

Equation (3.83) yields, for ε small enough,

$$\frac{1}{2}\int_{B(0,1)} |\nabla \hat{v}|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\hat{v}|^2)^2 \geq \pi |D_j|\,|\log\hat{\varepsilon}| + \mathcal{O}(1) = \pi |D_j| \log\frac{\rho}{\varepsilon} + \mathcal{O}(1),$$

and hence (3.82) holds. □

As in Lemma 3.11, we may compute an asymptotic expansion of $L_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$ in terms of vortices, which leads, in view of Lemma 3.26, to lower bounds for $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$:

Proof of Proposition 3.23: Let us consider the family of balls given in Proposition 3.10. As in the proof of Proposition 3.14, we can find $r_\varepsilon \in [R, (R+\sqrt{\rho_0})/2]$ such that (3.69) holds. Setting

$$I_R^+ = \big\{i,\ |p_i| > r_\varepsilon \text{ and } d_i \geq 0\big\} \quad \text{and} \quad I_R^- = \big\{i,\ |p_i| > r_\varepsilon \text{ and } d_i < 0\big\}, \tag{3.84}$$

we have $\overline{B}_i \subset \mathcal{D}_\varepsilon \setminus \overline{B}_{r_\varepsilon}$ for any $i \in I_R^+ \cup I_R^-$. By Propositions 3.10, 3.12, and 3.14, we infer that for ε small enough,

$$|v_\varepsilon| \geq \frac{1}{2} \quad \text{in } \Xi_\varepsilon := \mathcal{D}_\varepsilon \setminus \Bigg(\bigcup_{i\in I_R^+\cup I_R^-} B_i \cup \bigcup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)\Bigg).$$

Arguing exactly as in the proof of Lemma 3.11, we obtain that

$$\begin{aligned} L_\varepsilon(v_\varepsilon, \Xi_\varepsilon) = {}& \frac{-\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}^2(x_j^\varepsilon)\, D_j \\ & - \frac{\pi\Omega}{2}\sum_{i\in I_R^+\cup I_R^-} \big(\rho_{\mathrm{TF}}^2(p_i) - \nu_\varepsilon^2|\log\varepsilon|^{-3}\big)d_i + o_R(1). \end{aligned} \tag{3.85}$$

Recall (3.43); hence $L_\varepsilon(v_\varepsilon, \cup_{i\in I_R^+\cup I_R^-} B_i) = o(1)$. In the same way, we may prove that $L_\varepsilon(v_\varepsilon, \cup_{j=1}^{n_\varepsilon} B(x_j^\varepsilon, \rho)) = o(1)$. From Proposition 3.10 and (3.85), we deduce that

$$\begin{aligned} \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq {}& G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_R^+\cup I_R^-} B_i) + \sum_{i\in I_R^+\cup I_R^-} \frac{1}{2}\int_{B_i} \rho_{\mathrm{TF}}(x)|\nabla v_\varepsilon|^2 \\ & + L_\varepsilon(v_\varepsilon, \Xi_\varepsilon) + o_R(1) \\ \geq {}& G_\varepsilon(v_\varepsilon, B_R) - \frac{\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}^2(x_j^\varepsilon)\, D_j \\ & + \pi \sum_{i\in I_R^+\cup I_R^-} \rho_{\mathrm{TF}}(p_i)|d_i|\big(|\log\varepsilon| - \mathcal{K}_0\log|\log\varepsilon|\big) \\ & - \frac{\pi\Omega}{2}\sum_{i\in I_R^+\cup I_R^-} \big(\rho_{\mathrm{TF}}^2(p_i) - \nu_\varepsilon^2|\log\varepsilon|^{-3}\big)d_i + o_R(1). \end{aligned} \tag{3.86}$$

Since $p_i \notin \overline{B}_{r_\varepsilon}$ for $i \in I_R^+ \cup I_R^-$, we have $\rho_{\mathrm{TF}}(p_i) \ll \rho_0$ and we infer that for ε small enough,

$$\pi \sum_{i \in I_R^+ \cup I_R^-} \rho_{\mathrm{TF}}(p_i)|d_i|\big(|\log \varepsilon| - \mathcal{K}_0 \log|\log \varepsilon|\big) - \frac{\pi \Omega}{2} \sum_{i \in I_R^+ \cup I_R^-} \big(\rho_{\mathrm{TF}}^2(p_i) - v_\varepsilon^2 |\log \varepsilon|^{-3}\big) d_i$$

is nonnegative, which leads to

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq G_\varepsilon(v_\varepsilon, B_R) - \frac{\pi \Omega}{2} \sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}^2(x_j^\varepsilon)\, D_j + o_R(1). \tag{3.87}$$

Combining (3.80) and (3.87), we obtain (3.70). Similarly, the inequality (3.87) with $R = \sqrt{\rho_0}/2$ and (3.81) yields (3.72). □

3.6 Upper bound

We construct an upper bound with d vortices such that their rescaled position by $1/\sqrt{\Omega}$ minimizes the renormalized energy w introduced in (3.5). The main ingredients are taken from André and Shafrir [22] and have been applied to this problem by Ignat–Millot [81]. Given the splitting of the energy, to get an upper bound, we only need to get an upper bound for $G_\varepsilon(v)$.

Beforehand, we should recall a result in [33]: for $\tilde{\varepsilon} > 0$, consider

$$I(\tilde{\varepsilon}) = \min_{u \in \mathcal{C}} \frac{1}{2} \int_{B(0,1)} |\nabla u|^2 + \frac{1}{2\tilde{\varepsilon}^2}(1 - |u|^2)^2,$$

where

$$\mathcal{C} = \left\{ u \in H^1(B(0,1), \mathbf{C}),\ u(x) = \frac{x}{|x|} \text{ on } \partial B(0,1) \right\}.$$

Then we have

$$\lim_{\tilde{\varepsilon} \to 0} \big(I(\tilde{\varepsilon}) + \pi \log \tilde{\varepsilon} \big) = \gamma_0. \tag{3.88}$$

The expected test function v is going to be of modulus 1 except close to the vortex balls, where it will locally minimize I.

Proposition 3.27. *Let $d \geq 1$ be an integer. For any $\delta > 0$, there exists $\hat{v}_\varepsilon$ such that*

$$\limsup_{\varepsilon \to 0} \left\{ G_\varepsilon(\hat{v}_\varepsilon) + \frac{\pi}{2} \rho_0 d(\rho_0 \omega_1 - d + 1) \log|\log \varepsilon| \right\} \leq \min_{b \in \mathbf{R}^{2d}} w(b) + Q_d + \delta,$$

where w is defined by (3.5),

$$Q_d = \frac{\pi \rho_0}{2}(d^2 - d)\log 2 + \pi \rho_0 d \log \rho_0 + \rho_0 d \gamma_0 - \frac{\pi}{2} \rho_0 d^2, \tag{3.89}$$

and γ_0 is given by (3.88).

Proof: Step 1. Let $\sigma > 0$ and $\kappa > 0$ be two small parameters that we will choose later. We consider the function $\rho_\sigma : \overline{\mathcal{D}} \to \mathbf{R}$ given by

$$\rho_\sigma(x) = \begin{cases} \rho_{\mathrm{TF}}(x) & \text{if } |x| \le \sqrt{\rho_0 - \sigma}, \\ -2\sqrt{\rho_0-\sigma}\,|x| + 2\rho_0 - \sigma & \text{otherwise.} \end{cases}$$

It turns out that $\rho_\sigma \in C^1(\overline{\mathcal{D}})$, $\rho_\sigma \ge \rho_{\mathrm{TF}}$, and $\rho_\sigma \ge C\sigma^2$ in $\overline{\mathcal{D}}$ for some positive constant C. Since ρ_σ does not vanish in $\overline{\mathcal{D}}$, we may define $\Phi_\sigma : \mathcal{D} \to \mathbf{R}$ as the solution of the equation

$$\begin{cases} \operatorname{div}\,(\frac{1}{\rho_\sigma}\nabla\Phi_\sigma) = 2\pi d\,\delta_0 & \text{in } \mathcal{D}, \\ \Phi_\sigma = 0 & \text{on } \partial\mathcal{D}. \end{cases} \tag{3.90}$$

By the results in Chapter I of [33], there is a map $v_0^\sigma \in C^2(\overline{\mathcal{D}}\setminus\{0\}, S^1)$ satisfying

$$v_0^\sigma \wedge \nabla v_0^\sigma = \frac{1}{\rho_\sigma}\nabla^\perp\Phi_\sigma \quad \text{in } \mathcal{D}\setminus\{0\}. \tag{3.91}$$

Let $\Theta_{\kappa,\varepsilon} = \mathcal{D}\setminus B(0,\kappa^{-1}\Omega^{-1/2})$. By (3.90) and (3.91), for ε small enough, we have

$$\begin{aligned} \int_{\Theta_{\kappa,\varepsilon}} \rho_\sigma|\nabla v_0^\sigma|^2 = & \int_{\Theta_{\kappa,\varepsilon}} \frac{1}{\rho_\sigma}|\nabla\Phi_\sigma|^2 = -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{1}{\rho_{\mathrm{TF}}}\frac{\partial\Phi_\sigma}{\partial\nu}\Phi_\sigma \\ = & -\int_{\partial B(0,\kappa^{-1}\Omega^{-1/2})} \frac{\rho_0^2 d^2}{\rho_{\mathrm{TF}}}\left(\frac{\partial\Psi_\sigma}{\partial\nu} + \frac{1}{|x|}\right)\big(\Psi_\sigma + \log|x|\big), \end{aligned} \tag{3.92}$$

where $\Psi_\sigma(x) = (\rho_0 d)^{-1}\Phi_\sigma(x) - \log|x|$. Notice that $\Psi_\sigma \in C^{1,\alpha}(\overline{\mathcal{D}})$ for any $0 < \alpha < 1$, since it satisfies the equation

$$\begin{cases} \operatorname{div}\big(\frac{1}{\rho_\sigma}\nabla\Psi_\sigma\big) = f_\sigma(x) & \text{in } \mathcal{D}, \\ \Psi_\sigma = -\log|x| & \text{on } \partial\mathcal{D} \end{cases} \tag{3.93}$$

with

$$f_\sigma(x) = -\nabla\Big(\frac{1}{\rho_\sigma(x)}\Big)\cdot\frac{x}{|x|^2} = \begin{cases} \frac{-2}{\rho_\sigma^2(x)} & \text{if } |x| \le \sqrt{\rho_0-\sigma}, \\ \frac{-2\sqrt{\rho_0-\sigma}}{\rho_\sigma^2(x)|x|} & \text{otherwise.} \end{cases}$$

Since all functions are radial, Ψ_σ can be computed explicitly:

$$\Psi_\sigma(x) = -\int_{|x|}^{\sqrt{\rho_0}} \frac{\rho_{\mathrm{TF}}(t)}{t}\Big(\int_0^t f(s)s\,ds\Big)\,dt - \frac{1}{2}\log\rho_0.$$

In particular, we find that for $|x| < \sqrt{\rho_0-\sigma}$,

$$\Psi_\sigma(x) = \frac{1}{2\rho_0}(\rho_0 - |x|^2) - \frac{1}{2}\log\rho_0 + O(\sigma)).$$

It follows that as σ tends to 0, $\Psi_\sigma(0)$ tends to $\ell := \frac{1}{2} - \frac{1}{2}\log\rho_0$. From (3.92), we derive that

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}}\rho_{\mathrm{TF}}|\nabla v_0^\sigma|^2 - \pi\rho_0 d^2\log(\kappa\Omega^{1/2})\right\}$$
$$\leq \lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}}\rho_\sigma|\nabla v_0^\sigma|^2 - \pi\rho_0 d^2\log(\kappa\Omega^{1/2})\right\} \leq -\pi\rho_0 d^2\Psi_\sigma(0).$$

Consequently, we may choose σ small such that

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{\Theta_{\kappa,\varepsilon}}\rho_{\mathrm{TF}}|\nabla v_0^\sigma|^2 - \pi\rho_0 d^2\log(\kappa\Omega^{1/2})\right\} \leq -\pi\rho_0 d^2\ell + \frac{\delta}{2}. \tag{3.94}$$

Step 2. We are going to extend v_0^σ to $B(0,\kappa^{-1}\Omega^{-1/2})$. As in [33], we may write in a neighborhood of 0 (using polar coordinates)

$$v_0^\sigma(x) = \exp\big(i(d\theta + \psi_\sigma(x))\big),$$

where ψ_σ is a smooth function in that neighborhood. Let $(b_1,\ldots,b_d) \in \mathbf{R}^{2d}$ be a minimizing configuration for $w(\cdot)$, i.e.,

$$w(b_1,\ldots,b_d) = \min_{b\in\mathbf{R}^{2d}} w(b) \tag{3.95}$$

(note that we necessarily have $b_i \neq b_j$ for $i \neq j$). We choose κ sufficiently small such that $\max|b_j| \leq 1/(4\kappa)$ and we let $b_j^{(\varepsilon)} = \Omega^{-1/2}b_j$. Following the proof of Lemma 2.6 in [22], for $x \in A_{\kappa,\varepsilon} = B(0,\kappa^{-1}\Omega^{-1/2}) \setminus B(0,(2\kappa)^{-1}\Omega^{-1/2})$, we write

$$e^{i\psi_\sigma(0)}\prod_{j=1}^{d}\frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|} = \exp\big(i(d\theta + \phi_\varepsilon(x))\big),$$

where ϕ_ε is a smooth function satisfying $|\nabla\phi_\varepsilon(x)| \leq C_\sigma\,\kappa^2\Omega^{1/2}$ and $|\phi_\varepsilon(x) - \psi_\sigma(0)| = C_\sigma\,\kappa^2$ for $x \in A_{\kappa,\varepsilon}$. We define in $A_{\kappa,\varepsilon}$,

$$\hat{v}_\varepsilon(x) = \exp\big(i(d\theta + \hat{\psi}_\varepsilon(x))\big)$$

with

$$\hat{\psi}_\varepsilon(x) = \big(2 - 2\kappa\Omega^{1/2}|x|\big)\phi_\varepsilon(x) + \big(2\kappa\Omega^{1/2}|x| - 1\big)\psi_\sigma(x).$$

As in [22], we get, using (3.21),

$$\limsup_{\varepsilon\to 0}\Big\{G_\varepsilon(\hat{v}_\varepsilon, A_{\kappa,\varepsilon}) - \pi\rho_0 d^2\log 2\Big\}$$
$$\leq \limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{A_{\kappa,\varepsilon}}\rho_\sigma|\nabla\hat{v}_\varepsilon|^2 - \pi\rho_0 d^2\log 2\right\} \leq C_\sigma\,\kappa^2. \tag{3.96}$$

Next we define $\hat{v}_\varepsilon$ in $\Xi_{\kappa,\varepsilon} = B(0, (2\kappa)^{-1}\Omega^{-1/2}) \setminus \cup_{j=1}^{d} B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$ by

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)} \prod_{j=1}^{d} \frac{x - b_j^{(\varepsilon)}}{|x - b_j^{(\varepsilon)}|}.$$

Once more as in [22], we have, using (3.21),

$$\begin{aligned}\limsup_{\varepsilon\to 0} G_\varepsilon(\hat{v}_\varepsilon, \Xi_{\kappa,\varepsilon}) &\le \limsup_{\varepsilon\to 0} \frac{1}{2}\int_{\Xi_{\kappa,\varepsilon}} \rho_\sigma|\nabla\hat{v}_\varepsilon|^2 \\ &\le \pi\rho_0(d^2+d)\log\frac{1}{2\kappa} - \pi\rho_0\sum_{i\neq j}\log|b_i - b_j| + C_\sigma\,\kappa.\end{aligned} \tag{3.97}$$

Finally, in each $B_j^{(\varepsilon)} := B(b_j^{(\varepsilon)}, 2\kappa\Omega^{-1/2})$, we let

$$\hat{v}_\varepsilon(x) = e^{i\psi_\sigma(0)}\tilde{w}_\varepsilon^j\left(\frac{x - b_j^{(\varepsilon)}}{2\kappa\Omega^{-1/2}}\right), \tag{3.98}$$

where $\tilde{w}_\varepsilon^j$ achieves the minimum of

$$\frac{1}{2}\int_{B(0,1)} |\nabla v|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|v|^2)^2\,,\ \text{s.t. } v(y) = \prod_{i=1}^{d}\frac{2\kappa y + b_j - b_i}{|2\kappa y + b_j - b_i|}\ \text{ on } \partial B(0,1) \tag{3.99}$$

with

$$\hat{\varepsilon} = \frac{\varepsilon}{2\kappa\sqrt{\rho_0}\,\Omega^{-1/2}}.$$

As in the proof of Lemma 2.3 in [22], we derive

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B(0,1)}|\nabla\tilde{w}_\varepsilon^j|^2 + \frac{1}{2\hat{\varepsilon}^2}(1-|\tilde{w}_\varepsilon^j|^2)^2 - \pi|\log\hat{\varepsilon}|\right\} = \gamma_0 + X(\kappa),$$

where γ_0 is defined in (3.88) and $X(\kappa)$ denotes a quantity satisfying $X(\kappa)\to 0$ as $\kappa\to 0$. By scaling, we obtain

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}|\nabla\hat{v}_\varepsilon|^2 + \frac{\rho_0}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2 - \pi\log\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\} = \frac{\pi}{2}\log\rho_0 + \gamma_0 + X(\kappa).$$

Notice that in $B_j^{(\varepsilon)}$,

$$\rho_\sigma(x) = \rho_{\rm TF}(x) \le \rho_0 - (|\log\varepsilon| + \omega_1\log|\log\varepsilon|)^{-1}\min_{y\in B(b_j,2\kappa)}\frac{\rho_0|y|^2}{2},$$

and consequently,

$$\limsup_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{B_j^{(\varepsilon)}}\rho_\sigma|\nabla\hat{v}_\varepsilon|^2+\frac{\rho_0\rho_\sigma}{2\varepsilon^2}(1-|\hat{v}_\varepsilon|^2)^2-\pi\rho_0\log\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}$$
$$\leq\frac{\pi\rho_0}{2}\log\rho_0+\rho_0\gamma_0-\frac{\pi\rho_0|b_j|^2}{2}+X(\kappa).$$

This inequality and (3.21) yield

$$\limsup_{\varepsilon\to 0}\left\{G_\varepsilon(\hat{v}_\varepsilon,B_j^{(\varepsilon)})-\pi\rho_0\log\frac{2\kappa\Omega^{-1/2}}{\varepsilon}\right\}$$
$$\leq\frac{\pi\rho_0}{2}\log\rho_0+\rho_0\gamma_0-\frac{\pi\rho_0|b_j|^2}{2}+X(\kappa).\tag{3.100}$$

Combining (3.94), (3.96), (3.97), and (3.100), we conclude that for κ small enough,

$$\limsup_{\varepsilon\to 0}\left\{G_\varepsilon(\hat{v}_\varepsilon)-\pi\rho_0 d|\log\varepsilon|-\frac{\pi\rho_0}{2}(d^2-d)\log|\log\varepsilon|\right\}$$
$$\leq-\pi\rho_0\sum_{i\neq j}\log|b_i-b_j|-\frac{\pi\rho_0}{2}\sum_{j=1}^{d}|b_j|^2+Q_d+\delta.\tag{3.101}$$

Step 3. Now it remains to estimate $L_\varepsilon(\hat{v}_\varepsilon)$. By the results in Chapter IX in [33], for $\hat{\varepsilon}$ sufficiently small and each $j=1,\ldots,d$, there exists exactly one disc $\hat{D}_\varepsilon^j\subset B(0,1)$ with $\operatorname{diam}(\hat{D}_\varepsilon^j)\leq C\hat{\varepsilon}$ such that $|\tilde{w}_\varepsilon^j|\geq 1/2$ in $B(0,1)\setminus\hat{D}_\varepsilon^j$. By scaling, we infer that there exist exactly d discs $D_\varepsilon^1,\ldots,D_\varepsilon^d$ with $D_\varepsilon^j\subset B_j^{(\varepsilon)}$ and $\operatorname{diam}(D_\varepsilon^j)\leq C\varepsilon$ such that

$$|\hat{v}_\varepsilon|\geq\frac{1}{2}\quad\text{in }\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j.$$

We derive from (3.100) that

$$\left|L_\varepsilon(\hat{v}_\varepsilon,\cup_{j=1}^d D_\varepsilon^j)\right|\leq C\Omega\,\varepsilon\sum_{j=1}^{d}\left(G_\varepsilon(\hat{v}_\varepsilon,B_j^{(\varepsilon)})\right)^{1/2}\underset{\varepsilon\to 0}{\longrightarrow}0.$$

From (3.21), we infer that

$$\lim_{\varepsilon\to 0}\left|L_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)-L_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)\right|=0$$

and hence

$$\lim_{\varepsilon\to 0}\left|L_\varepsilon(\hat{v}_\varepsilon)-L_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)\right|=0.\tag{3.102}$$

To compute $L_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}\setminus\cup_{j=1}^d D_\varepsilon^j)$, we proceed as in the proof of Lemma 3.11 (here we use that $G_\varepsilon(\hat{v}_\varepsilon)\leq C|\log\varepsilon|$ by (3.101)). This yields

$$\lim_{\varepsilon\to 0}\left(L_\varepsilon(\hat{v}_\varepsilon,\mathcal{D}_\varepsilon\setminus\cup_{j=1}^d D_\varepsilon^j)+\frac{\pi\Omega}{2}\sum_{j=1}^{d}\rho_{\mathrm{TF}}^2(b_j^{(\varepsilon)})\right)=0,$$

since $\deg(\hat{v}_\varepsilon/|\hat{v}_\varepsilon|, \partial D_\varepsilon^j) = +1$ for $j = 1, \dots, d$. Expanding $\rho_{\mathrm{TF}}^2(b_j^{(\varepsilon)})$ and Ω, and setting $b_j = b_j^{(\varepsilon)}\Omega^{1/2}$, we deduce from (3.102) that

$$\lim_{\varepsilon\to 0}\left(L_\varepsilon(\hat{v}_\varepsilon) + \pi\rho_0 d\,|\log\varepsilon| + \frac{\pi}{2}\rho_0^2\omega_1 d\log|\log\varepsilon|\right) = \pi\rho_0\sum_{j=1}^{d}|b_j|^2. \qquad (3.103)$$

Combining (3.95), (3.101), and (3.103), we obtain the announced result. □

3.7 Final expansion and properties of vortices

We prove that vortices are of degree one and located close to the point of the maximum of ρ_{TF}.

3.7.1 Vortices have degree one

Lemma 3.28. *If ε is small enough, $D_j = +1$ for $j = 1, \dots, n_\varepsilon$.*

Proof. Recall that $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le o(1)$. This and (3.72) imply that

$$\pi\sum_{j=1}^{n_\varepsilon}|D_j|\,\rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} - \frac{\pi\rho_0\Omega}{2}\sum_{D_j>0}\rho_{\mathrm{TF}}(x_j^\varepsilon)\,D_j \le \mathcal{O}(1).$$

Since $\Omega = 2/\rho_0|\log\varepsilon| + o(|\log\varepsilon|)$, we derive that

$$\sum_{j=1}^{n_\varepsilon}|D_j|\,\rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} \le \sum_{D_j>0}D_j\,\rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\varepsilon| + o(|\log\varepsilon|).$$

Given that $\rho \ge \varepsilon^\mu$ and $D_j \ne 0$, it follows that

$$(1-\mu)\sum_{D_j<0}|D_j|\,\rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\varepsilon| \le \mu\sum_{D_j>0}|D_j|\,\rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\varepsilon| + o(|\log\varepsilon|).$$

By Proposition 3.12, $\rho_{\mathrm{TF}}(x_j^\varepsilon) \ge \rho_0/2$, and consequently,

$$\sum_{D_j<0}|D_j| \le \frac{2\mu}{1-\mu}\sum_{D_j>0}|D_j| + o(1) \le \frac{C\mu}{1-\mu} + o(1).$$

Choosing μ sufficiently small implies that $D_j > 0$ for $j = 1, \dots, n_\varepsilon$ whenever ε is small enough. Since $|x_j^\varepsilon| \le C$ and $D_j > 0$, we may now assert that

$$-\pi\sum_{i\ne j}D_i D_j\,\rho_{\mathrm{TF}}(x_j^\varepsilon)\log|x_i^\varepsilon - x_j^\varepsilon| \ge -C,$$

and thus $W_{\frac{\sqrt{\rho_0}}{2}} \geq -\pi \sum_{j=1}^{n_\varepsilon} D_j \Psi_{\frac{\sqrt{\rho_0}}{2},\varepsilon}(x_j^\varepsilon) - C$ is bounded below. Hence the inequality (3.70), applied with $R = \sqrt{\rho_0}/2$, together with $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \leq o(1)$ leads us to

$$\pi \sum_{j=1}^{n_\varepsilon} D_j^2\, \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\rho| + \pi \sum_{j=1}^{n_\varepsilon} D_j\, \rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} - \frac{\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}^2(x_j^\varepsilon)\, D_j \leq \mathcal{O}(1).$$

As previously, we derive from the expression of Ω,

$$\sum_{j=1}^{n_\varepsilon}(D_j^2 - D_j)\, \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\rho| \leq o(|\log\varepsilon|).$$

Since $\rho \leq \varepsilon^\mu$ and $\rho_{\mathrm{TF}}(x_j^\varepsilon) \geq \rho_0/2$, we conclude that

$$\frac{\mu\,\rho_0}{2}\sum_{j=1}^{n_\varepsilon}(D_j^2 - D_j) \leq o(1),$$

which yields $D_j = +1$ whenever ε is small enough. □

A direct consequence of Lemma 3.28 is the following improvement of Proposition 3.23:

Corollary 3.29. *For any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0}\,)$, we have*

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\varepsilon| - \frac{\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_i^\varepsilon, +1)\big) + \mathcal{O}_R(1).$$

Proof: The result follows directly from (3.70) and Lemma 3.28 that for any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0}\,)$,

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\varepsilon| - \frac{\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}^2(x_j^\varepsilon) + W_{R,\varepsilon}\big((x_i^\varepsilon, +1)\big) + \mathcal{O}_R(1).$$

On the other hand, we have proved in (3.33) that $|\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) - \mathcal{E}_{\eta_\varepsilon}(v_\varepsilon, \mathcal{D}_\varepsilon)| = o(1)$. Hence we have $\tilde{\mathcal{E}}_{\eta_\varepsilon}(v_\varepsilon) \geq \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1)$, and the conclusion follows. □

3.7.2 The subcritical case

We are now able to prove the rest of Theorem 3.1. It remains to prove the following proposition.

Proposition 3.30. *Assume that $\omega_1 < 0$. Then for ε sufficiently small, we have that*

$$|v_\varepsilon| \to 1 \quad \textit{in } L^\infty_{\mathrm{loc}}(\mathcal{D}) \textit{ as } \varepsilon \to 0. \tag{3.104}$$

Moreover,

$$\mathcal{E}_\varepsilon(v_\varepsilon) = o(1) \quad \textit{and} \quad G_\varepsilon(v_\varepsilon) = o(1). \tag{3.105}$$

Proof. We fix some $\frac{\sqrt{\rho_0}}{2} < R_0 < \sqrt{\rho_0}$. We have proved in (3.34) that $\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \le o(1)$, so that Corollary 3.29 with $R = \frac{\sqrt{\rho_0}}{2}$ leads to

$$\pi \sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log \varepsilon| - \frac{\pi\rho_0\Omega}{2} \sum_{j=1}^{n_\varepsilon} \rho_{\mathrm{TF}}(x_j^\varepsilon) \le C.$$

Since $\rho_{\mathrm{TF}}(x_j^\varepsilon) \ge \rho_0/2$ and $\omega_1 < 0$, we deduce that $\rho_0|\omega_1|\, n_\varepsilon \log|\log \varepsilon|$ is bounded and thus $n_\varepsilon \le o(1)$, which implies that $n_\varepsilon \equiv 0$ whenever ε is small enough. Using the notation (3.84), we derive from (3.86) that

$$\begin{aligned}\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \ge \pi &\sum_{i\in I_{R_0}^+\cup I_{R_0}^-} \rho_{\mathrm{TF}}(p_i)|d_i|\big(|\log \varepsilon| - \mathcal{K}_0 \log|\log \varepsilon|\big)\\ &- \frac{\pi\Omega}{2} \sum_{i\in I_{R_0}^+\cup I_{R_0}^-} \big(\rho_{\mathrm{TF}}^2(p_i) - \nu_\varepsilon^2|\log \varepsilon|^{-3}\big)d_i\end{aligned}$$

By (3.34), we have $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \le \mathcal{O}(|\log \varepsilon|^{-1})$. Since $\rho_{\mathrm{TF}}(p_i) \ll \rho_0$ for $i \in I_{R_0}^+ \cup I_{R_0}^-$, we infer that there exists $c > 0$ independent of ε such that

$$\begin{aligned}c \sum_{i\in I_{R_0}^+\cup I_{R_0}^-} \rho_{\mathrm{TF}}(p_i)|d_i||\log \varepsilon| &\le \pi \sum_{i\in I_{R_0}^+\cup I_{R_0}^-} \rho_{\mathrm{TF}}(p_i)|d_i|\big(|\log \varepsilon| - \mathcal{K}_0 \log|\log \varepsilon|\big)\\ &\quad - \frac{\pi\Omega}{2} \sum_{i\in I_{R_0}^+\cup I_{R_0}^-} \big(\rho_{\mathrm{TF}}^2(p_i) - \nu_\varepsilon^2|\log \varepsilon|^{-3}\big)d_i\\ &\le \mathcal{O}(|\log \varepsilon|^{-1}).\end{aligned}$$

Since $\rho_{\mathrm{TF}}(x) \ge |\log \varepsilon|^{-3/2}$ in $\mathcal{D}_\varepsilon$, we finally obtain

$$\sum_{i\in I_{R_0}^+\cup I_{R_0}^-} |d_i| \le \mathcal{O}(|\log \varepsilon|^{-1/2}).$$

Hence $\sum_{i\in I_{R_0}^+\cup I_{R_0}^-} |d_i| = 0$ for ε sufficiently small and we conclude from (3.85) that

$$L_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon \setminus \cup_{i\in I_{R_0}^+\cup I_{R_0}^-} B_i) = o(1).$$

By [80], we have $L_\varepsilon(v_\varepsilon, \cup_{i\in I_{R_0}^+\cup I_{R_0}^-} B_i) = o(1)$, so that $L_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = o(1)$. Consequently,

$$G_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) = \mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) + o(1) \le o(1).$$

Then the rest of the proof follows as in [80]. □

3.7.3 The supercritical case

In this section, we will prove Theorem 3.2. We assume that

$$2(d-1) < \omega_1\rho_0 < 2d \tag{3.106}$$

for some integer $d \ge 1$. We start by proving that in this regime, v_ε has vortices:

Proposition 3.31. *Assume that* (3.106) *holds. Then for ε sufficiently small, v_ε has exactly d vortices of degree one, i.e., $n_\varepsilon \equiv d$, and*

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) = -\frac{\pi}{2}\rho_0^2 d\,\omega_1 \log|\log \varepsilon| + \frac{\pi\rho_0}{2}(d^2 - d)\log|\log \varepsilon| + \mathcal{O}(1). \tag{3.107}$$

Proof. Step 1. We start by proving that $n_\varepsilon \geq 1$ for ε sufficiently small. By the upper bound of Proposition 3.27 (with $d = 1$), there exists $\hat{v}_\varepsilon$ such that

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \leq \mathcal{E}_{\eta_\varepsilon}(\hat{v}_\varepsilon) \leq -\frac{\pi}{2}\rho_0^2\omega_1\log|\log \varepsilon| + \mathcal{O}(1).$$

From here, it turns out by Corollary 3.29 with $R = \frac{\sqrt{\rho_0}}{2}$ (recall that $W_{\frac{\sqrt{\rho_0}}{2}} \geq \mathcal{O}(1)$), that

$$\begin{aligned}
-\frac{\pi}{2}\rho_0^2\omega_1\log|\log \varepsilon| + \mathcal{O}(1) &\geq \mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \geq \pi\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}(x_j^\varepsilon)|\log \varepsilon| - \frac{\pi\Omega}{2}\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}^2(x_j^\varepsilon)\\
&\geq \pi\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}(x_j^\varepsilon)\left(-\frac{\rho_0}{2}\omega_1\log|\log \varepsilon| + \frac{\Omega|x_j^\varepsilon|^2}{2}\right)\\
&\geq -\frac{\pi}{2}\rho_0^2\omega_1 n_\varepsilon\log|\log \varepsilon|.
\end{aligned}$$

Hence $n_\varepsilon \geq 1 + o(1)$ and the conclusion follows.

Step 2. Now we show that

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \geq -\frac{\pi}{2}\rho_0^2 n_\varepsilon\omega_1\log|\log \varepsilon| + \frac{\pi\rho_0}{2}(n_\varepsilon^2 - n_\varepsilon)\log|\log \varepsilon| + \mathcal{O}(1). \tag{3.108}$$

In the case $n_\varepsilon = 1$, we have already proved the result in the previous step. Then we may assume that $n_\varepsilon \geq 2$. Since $\left\|\Psi_{\frac{\sqrt{\rho_0}}{2},\varepsilon}\right\|_\infty = \mathcal{O}(1)$, we get from Corollary 3.29 with $R = \frac{\sqrt{\rho_0}}{2}$,

$$\begin{aligned}
\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) &\geq \pi\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}(x_j^\varepsilon)\Bigg(|\log \varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon}\log|x_i^\varepsilon - x_j^\varepsilon| - \frac{\Omega\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2}\Bigg) + \mathcal{O}(1)\\
&\geq \pi\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}(x_j^\varepsilon)\Bigg(-\frac{\rho_0}{2}\omega_1\log|\log \varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon}\log|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega|x_j^\varepsilon|^2}{2}\Bigg) + \mathcal{O}(1).
\end{aligned} \tag{3.109}$$

Since $\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \leq o(1)$, we derive that

$$-\sum_{i\neq j}\log|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|^2 \leq C\log|\log \varepsilon|.$$

On the other hand, $-\sum_{i\neq j}\log|x_i^\varepsilon - x_j^\varepsilon| \geq \mathcal{O}(1)$, so that

$$|x_j^\varepsilon|^2 \leq C\frac{\log|\log\varepsilon|}{|\log\varepsilon|}$$

and hence

$$\pi\sum_{j=1}^{n_\varepsilon}\rho_{\mathrm{TF}}(x_j^\varepsilon)\Bigg(-\frac{\rho_0}{2}\omega_1\log|\log\varepsilon| - \sum_{\substack{i=1\\ i\neq j}}^{n_\varepsilon}\log|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega|x_j^\varepsilon|^2}{2}\Bigg) \tag{3.110}$$

$$= -\frac{\pi}{2}\rho_0^2\, n_\varepsilon\omega_1\log|\log\varepsilon| - \pi\rho_0\sum_{i\neq j}\log|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi\rho_0\Omega}{2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|^2 + o(1).$$

Let $r = \max_j |x_j^\varepsilon|$. We remark that

$$-\sum_{i\neq j}\log|x_i^\varepsilon - x_j^\varepsilon| + \frac{\Omega}{2}\sum_{j=1}^{n_\varepsilon}|x_j^\varepsilon|^2 \geq -(n_\varepsilon^2 - n_\varepsilon)\log 2r + \frac{\Omega r^2}{2}$$

$$\geq \frac{n_\varepsilon^2 - n_\varepsilon}{2}\log|\log\varepsilon| + \mathcal{O}(1). \tag{3.111}$$

Combining (3.109), (3.110), and (3.111), we obtain (3.108).

Step 3. We are going to prove that $n_\varepsilon \geq d$. By Step 1, we may assume that $d \geq 2$. We use Proposition 3.27 to deduce that

$$\mathcal{E}_\varepsilon(v_\varepsilon) \leq -\frac{\pi}{2}\rho_0^2\, d\omega_1\log|\log\varepsilon| + \frac{\pi\rho_0}{2}(d^2 - d)\log|\log\varepsilon| + \mathcal{O}(1). \tag{3.112}$$

Matching (3.108) with (3.112), we deduce that

$$-\frac{\rho_0}{2}\omega_1 n_\varepsilon + \frac{n_\varepsilon^2 - n_\varepsilon}{2} \leq -\frac{\rho_0}{2}\omega_1 d + \frac{d^2 - d}{2} + o(1),$$

which yields

$$\frac{\rho_0}{2}\omega_1(d - n_\varepsilon) \leq \frac{(d - n_\varepsilon)(d + n_\varepsilon - 1)}{2} + o(1). \tag{3.113}$$

If we had $n_\varepsilon \leq d - 1$, it would lead to

$$(d-1) + \delta \leq \frac{d + n_\varepsilon - 1}{2} + o(1) \leq d - 1 + o(1),$$

which is impossible for ε small enough.

Assume now that $n_\varepsilon \geq d + 1$. As previously, we infer that (3.113) holds and therefore

$$d - \delta \geq \frac{d + n_\varepsilon - 1}{2} + o(1) \geq d + o(1),$$

which is also impossible for ε small. Hence $n_\varepsilon \equiv d$ whenever ε is small enough, which leads to (3.107) by (3.108) and (3.112). □

By Proposition 3.31, we may now assume that v_ε has exactly d vortices. The next lemma provides information on their location:

Lemma 3.32. *We have*

$$|x_j^\varepsilon| \le C|\log\varepsilon|^{-1/2}, \quad |x_i^\varepsilon - x_j^\varepsilon| \ge C|\log\varepsilon|^{-1/2} \quad \text{for } i \ne j.$$

Proof: Matching (3.107) with (3.109) and (3.110), we deduce that

$$-\pi\rho_0 \sum_{i\ne j} \log|x_i^\varepsilon - x_j^\varepsilon| + \frac{\pi\rho_0\Omega}{2}\sum_{j=1}^d |x_j^\varepsilon|^2 \le \pi\rho_0(d^2-d)\log\bigl(|\log\varepsilon|^{1/2}\bigr) + \mathcal{O}(1).$$

Hence

$$\sum_{j=1}^d \left(-\sum_{i\ne j}\log\left(\sqrt{|\log\varepsilon|}\,|x_i^\varepsilon - x_j^\varepsilon|\right) + \frac{\Omega|x_j^\varepsilon|^2}{2}\right) \le \mathcal{O}(1),$$

and the conclusion follows. □

Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ by Lemma 3.32, we may now improve the lower estimates obtained in Lemma 3.26 following the method in [143], proof of Proposition 5.2.

Lemma 3.33. *For any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0})$, we have*

$$G_\varepsilon(v_\varepsilon, B_R) \ge \pi\rho_0 \sum_{j=1}^d \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\varepsilon| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + \frac{\pi\rho_0 d}{2}\log\rho_0 + \rho_0 d\gamma_0 + o_R(1),$$

where γ_0 is given by (3.88).

Proof: Since $\frac{\rho}{|x_i^\varepsilon - x_j^\varepsilon|} = o(1)$ and $D_j = 1$, Proposition 3.25 yields

$$\frac{1}{2}\int_{\Theta_\rho} \rho_{\mathrm{TF}}(x)|\nabla v_\varepsilon|^2 \ge \pi\sum_{j=1}^d \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log\rho| + W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon) + o_R(1), \tag{3.114}$$

and it remains to estimate $G_\varepsilon(v_\varepsilon, B(x_j^\varepsilon,\rho))$ for $j = 1,\dots,d$. Since $D_j = 1$, we may write on $\partial B(x_j^\varepsilon,\rho)$ in polar coordinates with center x_j^ε,

$$v_\varepsilon(x) = |v_\varepsilon|(x)\, e^{i(\theta+\psi_j(\theta))}, \quad \theta\in[0,2\pi],$$

where $\psi_j \in H^1([0,2\pi],\mathbf{R})$ and $\psi_j(0) = \psi_j(2\pi) = 0$. Then in each disc $B(x_j^\varepsilon, 2\rho)$, we consider the map $\hat v_\varepsilon$ defined by

$$\begin{cases} v_\varepsilon(x) & \text{if } x\in B(x_j^\varepsilon,\rho),\\ \bigl(\frac{r-\rho}{\rho} + \frac{2\rho-r}{\rho}|v_\varepsilon|(x_j^\varepsilon + \rho\, e^{i\theta})\bigr)\exp i\left(\theta + \psi_j(\theta)\frac{2\rho-r}{\rho} + \psi_j(0)\frac{\rho-r}{\rho}\right) & \text{if not.}\end{cases}$$

Then $\hat{v}_\varepsilon = \exp i(\theta + \psi_j(0))$ on $\partial B(x_j^\varepsilon, 2\rho)$. Exactly as in the proof of Proposition 5.2 in [143], we prove that

$$\left| G_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho) \setminus B(x_j^\varepsilon, \rho)) - \pi\rho_{\mathrm{TF}}(x_j^\varepsilon)\log 2 \right| = o(1). \tag{3.115}$$

Since $|\rho_{\mathrm{TF}}(x) - \rho_{\mathrm{TF}}(x_j^\varepsilon)| = \mathcal{O}(\rho)$ on $B(x_j^\varepsilon, 2\rho)$, we may write

$$G_\varepsilon(\hat{v}_\varepsilon, B(x_j^\varepsilon, 2\rho)) = \frac{\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2} \int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 + o(1). \tag{3.116}$$

Since $\hat{v}_\varepsilon(x) = \frac{x - x_j^\varepsilon}{|x - x_j^\varepsilon|} e^{i\psi_j(0)}$ on $\partial B(x_j^\varepsilon, 2\rho)$, we obtain by scaling

$$\begin{aligned} \frac{1}{2} \int_{B(x_j^\varepsilon, 2\rho)} |\nabla \hat{v}_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2\varepsilon^2}(1 - |\hat{v}_\varepsilon|^2)^2 &\geq I\left(\frac{\varepsilon}{2\rho\sqrt{\rho_{\mathrm{TF}}(x_j^\varepsilon)}} \right) \\ &= \pi \log\frac{\rho}{\varepsilon} + \pi\log 2 + \frac{\pi}{2}\log\rho_{\mathrm{TF}}(x_j^\varepsilon) + \gamma_0 + o(1). \end{aligned}$$

With (3.115) and (3.116), we derive that for $j = 1, \dots, d$,

$$\begin{aligned} G_\varepsilon(v_\varepsilon, B(x_j^\varepsilon, \rho)) &\geq \pi\rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} + \frac{\pi\rho_{\mathrm{TF}}(x_j^\varepsilon)}{2}\log\rho_{\mathrm{TF}}(x_j^\varepsilon) + \rho_{\mathrm{TF}}(x_j^\varepsilon)\gamma_0 + o(1) \\ &\geq \pi\rho_{\mathrm{TF}}(x_j^\varepsilon)\log\frac{\rho}{\varepsilon} + \frac{\pi\rho_0}{2}\log\rho_0 + \rho_0\gamma_0 + o(1). \end{aligned}$$

Combining this estimate with (3.114), we get the result. □

We are now able to give the asymptotic expansion of $\mathcal{E}_\varepsilon(v_\varepsilon)$, which will allow us to locate precisely the vortices. This concludes the proof of Theorem 3.2.

Proposition 3.34. *Let $\tilde{x}_j^\varepsilon = \sqrt{\Omega}\, x_j^\varepsilon$ for $j = 1, \dots, d$. As $\varepsilon \to 0$, the configuration $\tilde{x}_j^\varepsilon$ tends to minimize the renormalized energy $w : \mathbf{R}^{2d} \to \mathbf{R}$ given by*

$$w(b_1, \dots, b_d) = -\pi\rho_0 \sum_{i \neq j} \log|b_i - b_j| + \frac{\pi\rho_0}{2} \sum_{j=1}^{d} |b_j|^2.$$

Moreover, we have

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) = -\frac{\pi\rho_0^2}{2} d\,\omega_1 \log|\log\varepsilon| + \frac{\pi\rho_0}{2}(d^2 - d)\log|\log\varepsilon| + \min_{b \in \mathbf{R}^{2d}} w(b) + Q_d + o(1), \tag{3.117}$$

where Q_d is given by (3.89).

Proof: From Lemma 3.33 and (3.87), we infer that for any $R \in [\sqrt{\rho_0}/2, \sqrt{\rho_0}\,)$,

$$\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon) \geq \pi \sum_{j=1}^{d} \rho_{\mathrm{TF}}(x_j^\varepsilon)|\log \varepsilon| - \frac{\pi\Omega}{2}\sum_{j=1}^{d}\rho_{\mathrm{TF}}^2(x_j^\varepsilon) + W_{R,\varepsilon} + \frac{\pi\rho_0 d}{2}\log \rho_0 + \rho_0 d\gamma_0 + o_R(1).$$

As in the proof of Corollary 3.29, this estimate also holds if $\mathcal{E}_\varepsilon(v_\varepsilon, \mathcal{D}_\varepsilon)$ is replaced by $\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)$. Expanding Ω and $\rho_{\mathrm{TF}}(x_j^\varepsilon)$, we derive that

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \geq \pi \sum_{j=1}^{d} \rho_{\mathrm{TF}}(x_j^\varepsilon)\bigg(- \frac{\rho_0}{2}\omega_1 \log|\log \varepsilon| + \frac{\Omega |x_j^\varepsilon|^2}{2}\bigg) + W_{R,\varepsilon} + \frac{\pi\rho_0 d}{2}\log \rho_0 + \rho_0 d\gamma_0 + o_R(1),$$

and Lemma 3.32 yields

$$\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \geq - \frac{\pi\rho_0^2}{2} d\, \omega_1 \log|\log \varepsilon| + \frac{\pi\rho_0}{2}\sum_{j=1}^{d}\Omega|x_j^\varepsilon|^2 + W_{R,\varepsilon} + \frac{\pi\rho_0 d}{2}\log \rho_0 + \rho_0 d\gamma_0 + o_R(1). \tag{3.118}$$

By Lemma 3.32, we also have

$$W_{R,\varepsilon} = -\pi\rho_0 \sum_{i\neq j}\log|x_i^\varepsilon - x_j^\varepsilon| - \pi \sum_{j=1}^{d}\Psi_{R,\varepsilon}(x_j^\varepsilon) + o(1). \tag{3.119}$$

Since $D_j = 1$ for all j, the function $\Psi_{R,\varepsilon}$ satisfies the equation

$$\begin{cases} \operatorname{div}\left(\frac{1}{\rho_{\mathrm{TF}}}\nabla\Psi_{R,\varepsilon}\right) = -\sum_{j=1}^{d} \rho_{\mathrm{TF}}(x_j^\varepsilon)\,\nabla\left(\frac{1}{\rho_{\mathrm{TF}}}\right)\cdot\nabla\left(\log|x - x_j^\varepsilon|\right) & \text{in } B_R,\\ \Psi_{R,\varepsilon} = -\sum_{j=1}^{d} \rho_{\mathrm{TF}}(x_j^\varepsilon)\log|x - x_j^\varepsilon| & \text{on } \partial B_R. \end{cases} \tag{3.120}$$

We infer from Lemma 3.32 that for $j = 1, \ldots, d$,

$$\rho_{\mathrm{TF}}(x_j^\varepsilon)\,\nabla\left(\frac{1}{\rho_{\mathrm{TF}}}\right)\cdot\nabla\left(\log|x - x_j^\varepsilon|\right) = \frac{-2\rho_0}{\rho_{\mathrm{TF}}^2(x)} + f_\varepsilon^j(x),$$

where f_ε^j satisfies $\|f_\varepsilon^j\|_{L^p(B_R)} = o_R(1)$ for any $p \in [1,2)$ and $\big\|\rho_0\log|x| - \rho_{\mathrm{TF}}(x_j^\varepsilon)\log|x - x_j^\varepsilon|\,\big\|_{C^1(\partial B_R)} = o(1)$. Let Ψ_R be the solution of

$$\begin{cases} \operatorname{div}\left(\frac{1}{\rho_{\mathrm{TF}}}\nabla\Psi_R\right) = \frac{-2}{\rho_{\mathrm{TF}}^2(x)} & \text{in } B_R,\\ \Psi_R = -\log|x| & \text{on } \partial B_R, \end{cases} \tag{3.121}$$

it follows from classical results that $\|\Psi_{R,\varepsilon} - \rho_0 d\Psi_R\|_{L^\infty(B_R)} = o_R(1)$. Hence we obtain from (3.119),

$$\lim_{\varepsilon\to 0}\left\{W_{R,\varepsilon}(x_1^\varepsilon,\dots,x_d^\varepsilon)+\pi\rho_0\sum_{i\neq j}\log|x_i^\varepsilon-x_j^\varepsilon|\right\}=-\pi\rho_0 d^2\Psi_R(0). \tag{3.122}$$

Equation (3.121) can be solved explicitly since it is a radial function, and one finds that Ψ_R converges to

$$\Psi(x)=\frac{1}{2\rho_0}(\rho_0-|x|^2)-\frac{1}{2}\log\rho_0.$$

It follows that as R tends to $\sqrt{\rho_0}$, $\Psi_R(0)$ tends to $\ell:=\frac{1}{2}-\frac{1}{2}\log\rho_0$.

Combining (3.118) and (3.122), we are led to

$$\begin{aligned}\liminf_{\varepsilon\to 0}&\left\{\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)+\frac{\pi\rho_0^2}{2}d\,\omega_1\log|\log\varepsilon|+\pi\rho_0\sum_{i\neq j}\log|x_i^\varepsilon-x_j^\varepsilon|-\frac{\pi\rho_0}{2}\sum_{j=1}^d\Omega|x_j^\varepsilon|^2\right\}\\&\geq\frac{\pi\rho_0 d}{2}\log\rho_0+\rho_0 d\gamma_0-\pi\rho_0 d^2\Psi_R(0).\end{aligned}$$

Let $\tilde{x}_j^\varepsilon=\sqrt{\Omega}\,x_j^\varepsilon$. Then

$$\begin{aligned}\liminf_{\varepsilon\to 0}&\left\{\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)+\frac{\pi\rho_0^2}{2}d\,\omega_1\log|\log\varepsilon|-\frac{\pi\rho_0}{2}(d^2-d)\log|\log\varepsilon|-w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)\right\}\\&\geq\frac{\pi\rho_0}{2}(d^2-d)\log(2)+\pi\rho_0 d\log\rho_0-\frac{\pi\rho_0 d^2}{2}\log\rho_0+\rho_0 d\gamma_0-\pi\rho_0 d^2\Psi_R(0).\end{aligned}$$

Since $\Psi_R(0)\to\ell$ as $R\to\sqrt{\rho_0}$, we conclude that the right-hand side tends to Q_d,

$$\begin{aligned}\liminf_{\varepsilon\to 0}&\left\{\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)+\frac{\pi\rho_0^2}{2}\omega_1 d\log|\log\varepsilon|-\frac{\pi\rho_0}{2}(d^2-d)\log|\log\varepsilon|-w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)\right\}\\&\geq Q_d,\end{aligned} \tag{3.123}$$

and hence

$$\begin{aligned}\liminf_{\varepsilon\to 0}&\left\{\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)+\frac{\pi\rho_0^2}{2}\omega_1 d\log|\log\varepsilon|-\frac{\pi\rho_0}{2}(d^2-d)\log|\log\varepsilon|\right\}\\&\geq\min_{b\in\mathbf{R}^{2d}}w(b)+Q_d.\end{aligned} \tag{3.124}$$

By the upper bound of Proposition 3.27, for any $\delta'>0$, there exists $\hat{v}_\varepsilon\in\mathcal{H}$ such that $\mathcal{E}(v_\varepsilon)\leq\mathcal{E}(\hat{v}_\varepsilon)$; hence

$$\begin{aligned}\limsup_{\varepsilon\to 0}&\left\{\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)+\frac{\pi}{2}\rho_0^2 d\,\omega_1\log|\log\varepsilon|-\frac{\pi\rho_0}{2}(d^2-d)\log|\log\varepsilon|\right\}\\&\leq\min_{b\in\mathbf{R}^{2d}}w(b)+Q_d+\delta'.\end{aligned} \tag{3.125}$$

Matching (3.124) with (3.125), we conclude that

$$\lim_{\varepsilon\to 0}\left\{\mathcal{E}_{\eta_\varepsilon}(v_\varepsilon)+\frac{\pi}{2}\rho_0^2 d\,\omega_1\log|\log\varepsilon|-\frac{\pi\rho_0}{2}(d^2-d)\log|\log\varepsilon|\right\}=\min_{b\in\mathbf{R}^{2d}} w(b)+Q_d$$

since δ' is arbitrarily small. Returning to (3.123), we are led to

$$\min w(b)+Q_d-\limsup_{\varepsilon\to 0} w(x_1^\varepsilon,\dots,x_d^\varepsilon)\geq Q_d$$

and therefore $\lim_{\varepsilon\to 0} w(\tilde{x}_1^\varepsilon,\dots,\tilde{x}_d^\varepsilon)=\min_{b\in\mathbf{R}^{2d}} w(b)$, which ends the proof. □

3.8 Open Questions

3.8.1 Vortices in the region of low density

As we mentioned in the introduction, the techniques introduced in this chapter do not allow us to investigate the existence of vortices in the region of low density. For Ω sufficiently below the critical velocity, we believe that there are no vortices at all, as stated in Open Problem 3.1.

Open Problem 3.2 *For Ω such that $\lim_{\varepsilon\to 0}\Omega/|\log\varepsilon|>\omega_0^*$, are all the vortices close to the origin or are there some located in $\mathcal{D}\setminus\mathcal{D}_\varepsilon$?*

When the minimization is set in $\mathbf{R}^2$ instead of $\mathcal{D}$, the results of this chapter have been proved in [80, 81]. The same open problems as 3.1 and 3.2 can be stated. An intermediate result would be to prove that the vortices lie in a bounded domain.

3.8.2 Other trapping potentials

Open Problem 3.3 *Address the minimization of (3.1) when $\rho_{\mathrm{TF}}(\mathbf{r})$ is still a radial function such that the domain $\mathcal{D}=\{\rho_{\mathrm{TF}}>0\}$ is a disc.*

Let us call $\xi(r)$ the primitive that vanishes on $\partial\mathcal{D}$ of $-r\rho_{\mathrm{TF}}(r)$. If $\rho_{\mathrm{TF}}(r)$ is decreasing and the maximum of $\xi(r)/\rho_{\mathrm{TF}}(r)$ is achieved at the origin, then a similar proof to the one presented in this chapter should hold and vortices should appear close to the origin.

On the other hand, if the maximum of $\xi(r)/\rho_{\mathrm{TF}}(r)$ is achieved for $r=r_0>0$, then vortices should appear on the circle of radius r_0, as illustrated in Figure 1.6. The critical velocity for the existence of n vortices should be of order $\omega_0^*|\log\varepsilon|+\omega_n$, where ω_n is of order 1, instead of order $\log|\log\varepsilon|$. The main difficulty in the proof relies on the estimate (3.50), which is not easy to get when the vortex balls do not approach a single point but a curve. Once this is proved, the machinery of Section 3.4 can be used to derive the refined structure of vortices. A totally new feature is that when there are n vortices in the system, they are no longer at distance $1/\sqrt{|\log\varepsilon|}$, but at distance of order 1. The logarithm involved in the renormalized energy w should be replaced by something else. Some similar features will be addressed in the next chapter for the quartic potential.

3.8.3 Intermediate Ω

Open Problem 3.4 *Let $\omega_0 = \lim_{\varepsilon\to 0} \Omega/|\log \varepsilon|$. Let u_ε be a minimizer of E_ε, and η_ε be as defined in Proposition 3.3. Then*

$$\lim_{\varepsilon\to 0} \frac{E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon)}{\Omega^2} = \min J - \frac{1}{2}\int_{\mathcal{D}} r^2 \rho_{\mathrm{TF}}, \tag{3.126}$$

where

$$J(w) = \frac{1}{2\omega_0}\int_{\mathcal{D}} \rho_{\mathrm{TF}} \left|\operatorname{div}\left(\frac{1}{\rho_{\mathrm{TF}}}\nabla w\right) + 2\right| + \frac{1}{2}\int_{\mathcal{D}} \frac{1}{\rho_{\mathrm{TF}}}|\nabla w|^2 \tag{3.127}$$

is defined for $w \in H_0^1(\mathcal{D})$ such that

$$\int_{\mathcal{D}} \frac{1}{\rho_{\mathrm{TF}}}|\nabla w|^2 < \infty \textit{ and } \operatorname{div}\left(\frac{1}{\rho_{\mathrm{TF}}}\nabla w\right) + 2 \textit{ is a Radon measure in } \mathcal{D}.$$

The proof requires the use of $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$ and the energy splitting (3.10), but we write the two terms including the gradient as a perfect square, which yields

$$E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon) = \int_{\mathcal{D}} \frac{\eta_\varepsilon^2}{2}|\nabla v_\varepsilon - i\Omega \times r v_\varepsilon|^2 - \frac{\eta_\varepsilon^2}{2}\Omega^2 r^2 |v_\varepsilon|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2. \tag{3.128}$$

Minimizing this on v_ε allows us to get two equations, on the modulus and phase of v_ε, the second one being

$$\operatorname{div}\,(\eta_\varepsilon^2(iv_\varepsilon, \nabla v_\varepsilon) - \Omega\eta_\varepsilon^2 \mathbf{r}^\perp) = 0. \tag{3.129}$$

This implies that there exists w_ε satisfying

$$\nabla^\perp w_\varepsilon = \eta_\varepsilon^2(iv_\varepsilon, \nabla v_\varepsilon) - \Omega\eta_\varepsilon^2 \mathbf{r}^\perp. \tag{3.130}$$

One should prove that w_ε/Ω converges weakly in H_0^1 to w_*, the unique minimizer of J that is a solution of the following free boundary problem:

$$\begin{cases} (\operatorname{div}\,(1/(\rho_{\mathrm{TF}})\nabla w_*) + 2)\,(w_* - \rho_{\mathrm{TF}}/(2\omega_0)) = 0 \text{ in } \mathcal{D}, \\ w_* = 0 \text{ on } \partial\mathcal{D}, \\ w_* \le \rho_{\mathrm{TF}}/(2\omega_0) \text{ in } \mathcal{D}, \\ \operatorname{div}\,(1/(\rho_{\mathrm{TF}})\nabla w_*) + 2 \ge 0 \text{ in } \mathcal{D}. \end{cases}$$

The measure $\mu_* = \operatorname{div}\,(1/(\rho_{\mathrm{TF}})\nabla w_*) + 2$ is the vortex density. It is supported in $\mathcal{D}_* = \{x \in \mathcal{D},\ w_* = \rho_{\mathrm{TF}}/(2\omega_0)\}$. This domain is nonempty, that is, vortices start to exist as soon as $\omega_0 \ge \omega_0^* = 2/\rho_0$. This critical value is consistent with the one found in Theorem 3.1. The region $\mathcal{D}_*$ corresponds to the region where there is a uniform distribution of vortices (it is an inner disc), while in the exterior of $\mathcal{D}_*$, defined by $\{\rho_{\mathrm{TF}} \le 2/\omega_0\}$, there are no vortices. Due to the special shape of ρ_{TF}, it turns out that the solution of $\operatorname{div}\,(1/(\rho_{\mathrm{TF}})\nabla w_*) = -2$ in $\mathcal{D}$ with 0 boundary condition is $\rho_{\mathrm{TF}}^2/4$. We also expect the vortex density $\mu_\varepsilon = (2\pi/\Omega)\sum_i d_i \delta_{p_i}$, where p_i are the centers of the vortex balls, to converge to μ_*.

More detailed results in the spirit of [135] are certainly possible.

3.8.4 Time-dependent problem

Open Problem 3.5 *Consider an initial vortex-free solution of the time-dependent problem with Schrödinger dynamics and analyze the evolution equation for fixed positive* Ω*. The final state should be close to a one-vortex solution if* Ω *is appropriate.*

4

Other Trapping Potentials

In this chapter, we are interested in the minimizers of the energy

$$E_\varepsilon(u) = \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla u|^2 - \Omega \mathbf{r}^\perp \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2}\left(|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r})\right)^2 \right\} dx\, dy, \quad (4.1)$$

for various functions $\rho_{\mathrm{TF}}(\mathbf{r})$. As before, $\mathbf{r} = (x, y)$, $\mathbf{r}^\perp = (-y, x)$, $(iu, \nabla u) = i(\bar{u}\nabla u - u\nabla\bar{u})/2$, ε is a small parameter, and Ω is the given rotational velocity. We assume that $\mathcal{D} = \{\rho_{\mathrm{TF}} > 0\}$ and $\rho_{\mathrm{TF}}(\mathbf{r})$ describes respectively a nonradial harmonic confinement and a quartic trapping potential, that is, the model cases are

$$\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - x^2 - \alpha^2 y^2 \text{ with } \alpha \neq 1 \text{ and } \rho_0 \text{ s.t. } \int_{\mathcal{D}} \rho_{\mathrm{TF}} = 1, \quad (4.2)$$

$$\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 + (b-1)r^2 - (k/4)r^4 \text{ and } \rho_0 \text{ s.t. } \int_{\mathcal{D}} \rho_{\mathrm{TF}} = 1. \quad (4.3)$$

In case (4.3), for certain values of b and k, the domain $\mathcal{D}$ becomes an annulus, and this changes the pattern of vortices.

Both cases are motivated by experiments: the first one corresponds to the real harmonic potential of the experiments [107, 108], since it is never exactly radial but bears some inhomogeneity. The second case is motivated by recent ENS experiments [40, 150] in which an extra laser beam is added to the system and thus modifies the trapping potential so that a giant vortex can be observed (see for instance Figure 1.4, (f); the hazy region in the center corresponds to a hole).

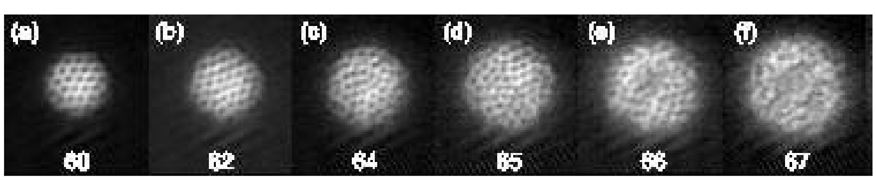


Fig. 4.1. Experimental vortices for a potential of type (4.3). Courtesy of V. Bretin and J. Dalibard.

In the first section, we study case (4.2): the results and techniques are very similar to those of the previous chapter, except that now, the critical velocity and the renormalized energy depend on α. The proofs have been done by Ignat and Millot [80, 81]. The main difference with the radial case relies on the fact that the vortex-free solution η_ε has a globally defined phase. Thus, in making the splitting of the energy, the test function is going to be modified to include this global phase.

The next section is devoted to the second model case of ρ_{TF}, where $\mathcal{D}$ is an annulus. The topology of $\mathcal{D}$ implies that the order of the critical velocities changes: there a first critical velocity of order 1 above which the minimizer has a degree on any circle contained in the annulus. This degree is due to the presence of a giant vortex in the central ball. Then for velocities of order $\omega_0|\log\varepsilon|+\omega_1$, the giant vortex has a circulation of order $|\log\varepsilon|$. If $\omega_0 < \omega_0^*$, there are no vortices in the annulus, while for $\omega_0 > \omega_0^*$, the number of vortices in the annulus depends on ω_1 and they are arranged on a specific circle (as illustrated in Figure 1.6), which we are able to characterize.

4.1 Non radial harmonic potential

The results of Chapter 3 can be extended when ρ_{TF} is given by (4.2), with ω_0^* and ω_1^n now depending on α. As before, we will have to restrict our analysis to $\mathcal{D}_\delta = \{\mathbf{r} \in \mathcal{D},\ \mathrm{dist}(|\mathbf{r}|^2, \partial\mathcal{D}) > \delta\}$.

Theorem 4.1. *We assume a specific asymptotic form for the rotation Ω,*

$$\Omega = \omega_0|\log\varepsilon| + \omega_1 \log|\log\varepsilon|. \tag{4.4}$$

Let u_ε be a sequence of minimizers of E_ε in $H_0^1(\mathcal{D})$. Then $\omega_0^* = (1+\alpha^2)/\rho_0$ is a critical value in the following sense:

(i) If $\omega_0 < \omega_0^*$, or if $\omega_0 = \omega_0^*$ and $\omega_1 < 0$, for any $\delta > 0$, if ε is smaller than some ε_δ, then u_ε does not vanish in $\mathcal{D}_\delta$. In addition, as ε tends to 0, $|u_\varepsilon|$ converges to $\sqrt{\rho_{\mathrm{TF}}}$ in $L^\infty_{\mathrm{loc}}(\mathcal{D})$, and

$$E_\varepsilon(u_\varepsilon) = \mathcal{E}(\varepsilon) + o(1), \tag{4.5}$$

where $\mathcal{E}(\varepsilon)$ does not depend on u_ε or Ω.

(ii) If $\omega_1^n < \omega_1 < \omega_1^{n+1}$, with $\omega_1^n = (1+\alpha^2)(n-1)/\rho_0$, for any $\delta > 0$, if ε is smaller than some ε_δ, u_ε has exactly n vortices p_i^ε of degree one in $\mathcal{D}_\delta$. Moreover,

$$|p_i^\varepsilon| < C/\sqrt{\Omega} \quad \text{for any} \quad i, \quad \text{and} \quad |p_i^\varepsilon - p_j^\varepsilon| > C/\sqrt{\Omega},$$

where C is independent of ε. Let $\tilde{p}_i^\varepsilon = p_i^\varepsilon/\sqrt{\Omega}$. Then the configuration $\tilde{p}_i^\varepsilon$ tends to minimize the energy w defined in $\mathbf{R}^{2n}$ by

$$w(b_1, \ldots, b_n) = -\pi\rho_0 \sum_{i\neq j} \log|b_i - b_j| + \frac{\pi\rho_0}{1+\alpha^2} \sum_i |b_i|_\alpha^2, \tag{4.6}$$

where $|\mathbf{r}|_\alpha^2 = x^2 + \alpha^2 y^2$.*We have the following asymptotic expansion for the energy:*

$$E_\varepsilon(u_\varepsilon) = \mathcal{E}(\varepsilon) + \frac{\pi}{1+\alpha^2}(\omega_0^* - \omega_0)|\log \varepsilon|$$
$$+n\frac{\rho_0}{2}\left(n - 1 - \frac{2}{1+\alpha^2}\omega_1\rho_0\right)\log|\log\varepsilon| + \min_{\mathbf{R}^{2n}} w + C_{n,\alpha} + o(1), \quad (4.7)$$

where $\mathcal{E}(\varepsilon)$ *does not depend on* u_ε*, and* $C_{n,\alpha}$ *is an explicit constant that depends only on* n *and* α*.*

We will not include a complete proof here, since it is very similar to that of the previous chapter. We are only going to point out the major differences and refer to [80, 81] for details. In order to get the splitting of the energy, we introduce the vortex-free minimizer $f_\varepsilon e^{iS_\varepsilon}$ of the energy E_ε, where f_ε does not vanish in $\mathcal{D}$. Its phase is thus globally defined. Given the formulation of E_ε, we find that $(f_\varepsilon, S_\varepsilon)$ minimize

$$F_\varepsilon(f,S) = \int_{\mathcal{D}} \frac{1}{2}|\nabla f|^2 + \frac{1}{2}f^2|\nabla S|^2 - \Omega f^2\mathbf{r}^\perp\cdot\nabla S + \frac{1}{4\varepsilon^2}\left(f^2 - \rho_{\mathrm{TF}}(\mathbf{r})\right)^2. \quad (4.8)$$

One can prove, as in the previous chapter for η_ε, that f_ε^2 converges to ρ_{TF}. The limit S of S_ε is more involved to obtain. Formal computations yield that $S = \Omega(\alpha^2-1)xy/(\alpha^2+1)$. Instead of using $\eta_\varepsilon = f_\varepsilon e^{iS_\varepsilon}$ to define the ratio $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, we use

$$\eta_\varepsilon = g_\varepsilon e^{iS}, \quad \text{where} \quad S = \Omega\frac{\alpha^2-1}{\alpha^2+1}xy$$

and g_ε minimizes $E_\varepsilon(f) = F_\varepsilon(f,0)$ among real-valued functions f. For $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, we obtain the following splitting of energy:

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_{\eta_\varepsilon}(v_\varepsilon) \text{ where } \mathcal{E}_{\eta_\varepsilon}(v) = G_{\eta_\varepsilon}(v) + L_{\eta_\varepsilon}(v) + R_{\eta_\varepsilon}(v) \quad (4.9)$$

and

$$G_{\eta_\varepsilon}(v) = \int_{\mathcal{D}} \frac{|\eta_\varepsilon|^2}{2}|\nabla v|^2 + \frac{|\eta_\varepsilon|^4}{4\varepsilon^2}(|v|^2-1)^2, \quad (4.10)$$
$$L_{\eta_\varepsilon}(v) = \frac{\Omega}{\alpha^2+1}\int_{\mathcal{D}} |\eta_\varepsilon|^2\nabla^\perp\rho_{\mathrm{TF}}\cdot(iv,\nabla v), \quad (4.11)$$
$$R_{\eta_\varepsilon}(v) = \frac{1}{2}\int_{\mathcal{D}} |\eta_\varepsilon|^2(|v|^2-1)(|\nabla S|^2 - 2\Omega\mathbf{r}^\perp\cdot\nabla S). \quad (4.12)$$

Let us justify this expansion: since g_ε minimizes E_ε among real-valued functions, it is a solution of an equation that we can multiply by $(|ve^{iS}|^2 - 1)$ and integrate. We obtain as in the proof of Lemma 3.7,

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(g_\varepsilon) + G_{\eta_\varepsilon}(v_\varepsilon e^{iS}) - \int_{\mathcal{D}} |\eta_\varepsilon|^2\Omega\mathbf{r}^\perp\cdot(iv_\varepsilon e^{iS}, \nabla(v_\varepsilon e^{iS}))$$

$$= E_\varepsilon(g_\varepsilon e^{iS}) + G_{\eta_\varepsilon}(v_\varepsilon) + R_{\eta_\varepsilon}(v_\varepsilon) - \int_{\mathcal{D}} |\eta_\varepsilon|^2 (\Omega \mathbf{r}^\perp - \nabla S) \cdot (iv_\varepsilon, \nabla v_\varepsilon).$$

Because of the specific shape of the trapping potential ρ_{TF}, it turns out that $\Omega\mathbf{r}^\perp - \nabla S = -\Omega\nabla^\perp\rho_{\mathrm{TF}}/(\alpha^2+1)$, and thus (4.9) holds. If ρ_{TF} were another trapping potential, S would be chosen to minimize (4.8) with $f = \rho_{\mathrm{TF}}$. Thus it satisfies div $(\rho_{\mathrm{TF}}(\nabla S - \Omega\mathbf{r}^\perp)) = 0$. This implies that there is a potential function ξ such that $\rho_{\mathrm{TF}}(\nabla S - \Omega\mathbf{r}^\perp) = -\Omega\nabla^\perp\xi$. In the case of the harmonic potential, ξ is explicit and is equal to $-\rho_{\mathrm{TF}}^2/2(1+\alpha^2)$. In other cases, it is not explicit. As pointed out in the open problems of the previous chapter, the important property in carrying out our proof is in fact that $\max \xi/\rho_{\mathrm{TF}}$ is reached at the origin.

The first two terms G_{η_ε} and L_{η_ε} are very similar to the case $\alpha = 1$ and are estimated similarly. For the remainder term R_{η_ε}, we can prove that it tends to 0 as ε tends to 0, using the Cauchy–Schwarz inequality and the fact that $\int_{\mathcal{D}} |\eta_\varepsilon|^4(|v|^2-1)^2$ is bounded by $C\varepsilon^2|\log\varepsilon|^2$. Let us insist on the fact that it is important to use η_ε or $g_\varepsilon e^{iS}$ rather than just $|\eta_\varepsilon|$ in the splitting of the energy.

The rest of the estimates follow the same lines as the case $\alpha = 1$, with the adaptations $|\mathbf{r}|_\alpha^2 = x^2+\alpha^2y^2$ replaces the usual norm and the primitive ξ of $-r\rho_{\mathrm{TF}}(\mathbf{r})$ is now $\rho_{\mathrm{TF}}^2/2(1+\alpha^2)$.

4.2 Quartic potential

In this section, we assume that $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 + (b-1)r^2 - (k/4)r^4$, ρ_0 is such that $\int_{\mathcal{D}} \rho_{\mathrm{TF}} = 1$, and $b > 1 + (3k^2/4)^{1/3}$. Thus the domain $\mathcal{D} = \{\rho_{\mathrm{TF}} > 0\}$ is an annulus. The results are based on [3]. We are going to consider first the case when Ω is of order 1, which provides a giant vortex, and then the case when Ω is of order $|\log\varepsilon|$, for which there are also vortices arranged on a circle.

4.2.1 Giant vortex

We are going to determine the critical velocities for which the circulation on any circle contained in $\mathcal{D}$ gets bigger than d. This corresponds to the existence of a giant vortex located in the central hole of the annulus.

As before, the energy E_ε splits into two parts, the energy of the density profile and a reduced energy of the complex phase, which allows us to compute the limiting energy and identify the size of the giant vortex. Let us introduce the density profile: it is the minimizer of the energy E_ε when $\Omega = 0$. When $\Omega = 0$, $E_\varepsilon(\eta) = F_\varepsilon(\eta)$, where

$$F_\varepsilon(\eta) = \int_{\mathcal{D}} \frac{1}{2}|\nabla\eta|^2 + \frac{1}{4\varepsilon^2}(|\eta|^2 - \rho_{\mathrm{TF}}(r))^2. \tag{4.13}$$

Properties of the minimizer η_ε are similar to those proved in Proposition 3.3. For instance, η_ε is real-valued, and η_ε^2 converges to ρ_{TF} in $L^2(\mathcal{D})$ and uniformly on any compact set.

Theorem 4.2. *Let*

$$g_0(d) = \frac{1}{2}\Lambda_1 d^2 - \Omega d,\ d \in \mathbf{Z}, \quad \textit{where } \Lambda_1 = \int_{\mathcal{D}} \frac{\rho_{\mathrm{TF}}(r)}{r^2}. \tag{4.14}$$

Let $\Omega_d = \Lambda_1(d - 1/2)$ for $d \geq 1$ and $\Omega_0 = 0$. If u_ε is a sequence of minimizers of E_ε, and $\Omega_d \leq \Omega < \Omega_{d+1}$, then:

(i) $E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon) \to g_0(d)$, as ε tends to 0.
(ii) There exists a subsequence $\varepsilon \to 0$ and $\alpha \in \mathbf{C}$ with $|\alpha| = 1$ such that

$$\frac{u_\varepsilon}{\eta_\varepsilon} \to \alpha e^{id\theta} \ \textit{ in } H^1_{\mathrm{loc}}(\mathcal{D}), \textit{ and } \quad \left|\frac{u_\varepsilon}{\eta_\varepsilon}\right| \to 1, \ \textit{ locally uniformly in } \mathcal{D}.$$

(iii) For every fixed r such that $\partial B_r(0) \subset \mathcal{D}$, $\deg(\frac{u_\varepsilon}{\eta_\varepsilon}, \partial B_r) = d$ for ε sufficiently small.

The circulation of the giant vortex is thus equal to d and we find that the critical velocity for having a giant vortex with circulation d is proportional to d.

The proof relies on the splitting of the energy as previously:

$$E_\varepsilon(u) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_{\eta_\varepsilon}(v), \quad \text{where } \mathcal{E}_{\eta_\varepsilon}(v) = G_{\eta_\varepsilon}(v) + L_{\eta_\varepsilon}(v) \tag{4.15}$$

and

$$G_{\eta_\varepsilon}(v) = \int_{\mathcal{D}} \frac{\eta_\varepsilon^2}{2}|\nabla v|^2 + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2 - 1)^2, \tag{4.16}$$

$$L_{\eta_\varepsilon}(v) = -\int_{\mathcal{D}} \eta_\varepsilon^2 \Omega \mathbf{r}^\perp \cdot (iv, \nabla v)\, dx\, dy. \tag{4.17}$$

We need to understand the contributions of the main terms. As before, G_{η_ε} will force v_ε to have a modulus close to 1. We are going to prove that $\mathcal{E}_\varepsilon(v_\varepsilon)$ tends to $g_0(d)$. The following lemma provides an indication for a lower bound for the energy.

Lemma 4.3. *For $v \in H^1(\mathcal{D}, S^1)$, let*

$$G_0(v) = \int_{\mathcal{D}} \frac{\rho_{\mathrm{TF}}}{2}|\nabla v|^2 - \rho_{\mathrm{TF}}\Omega \mathbf{r}^\perp \cdot (iv, \nabla v). \tag{4.18}$$

Then for any r such that $\partial B_r \subset \mathcal{D}$,

$$G_0(v) \geq g_0(d), \ \textit{where } \ d = \deg(v, \partial B_r),$$

and g_0 is given in (4.14). In particular, the minimum of G_0 in $H^1(\mathcal{D}, S^1)$ is achieved, and any minimizer has the form $v_0 = \alpha e^{iD_0\theta}$, where D_0 minimizes g_0 in $\mathbf{Z}$ and $\alpha \in \mathbf{C}$ is a constant with $|\alpha| = 1$.

Proof: Let v be a smooth function in $\mathcal{D}$ with values in S^1. The degree of v is constant on any concentric circle $S_r \subset \mathcal{D}$ and we call it d. We define the energy of v on the circle S_r by

$$e(r; v) = \int_{S_r} \frac{\rho_{\mathrm{TF}}}{2} |\nabla v|^2 - \rho_{\mathrm{TF}} \Omega \mathbf{r}^\perp \cdot (iv, \nabla v).$$

First, we note that

$$\int_{S_r} \rho_{\mathrm{TF}} \Omega \mathbf{r}^\perp \cdot (iv, \nabla v) = 2\pi r \rho_{\mathrm{TF}}(r) \Omega d$$

depends only on the homotopy class of v. For the other term, the Cauchy–Schwarz inequality implies

$$\int_{S_r} |\nabla v|^2 \geq \int_{S_r} (iv, \nabla v)^2 \geq \frac{1}{2\pi r} \left(\int_{S_r} (iv, \nabla v) \cdot \tau \right)^2 = \frac{2\pi d^2}{r}. \tag{4.19}$$

Therefore, we have the lower bound

$$e(r; v) \geq 2\pi \rho_{\mathrm{TF}}(r) \left(\frac{1}{2} \frac{d^2}{r} - \Omega d r \right) = e\left(r; e^{id\theta}\right) \tag{4.20}$$

for any $v \in H^1(\mathcal{D}, S^1)$ and for almost every $r \in (r_0, R_0)$. Integrating (4.20) over r (recall that we assume $\int_{\mathcal{D}} \rho_{\mathrm{TF}}(r)\, dx = 1$) we obtain $G_0(v) \geq G_0(e^{id\theta}) = g_0(d)$.

Now let v be chosen to minimize G_0:

$$g_0(d) = G_0(e^{id\theta}) \leq G_0(v) \leq G_0(e^{iD_0\theta}) = g_0(D_0).$$

Since D_0 minimizes g_0 over $\mathbf{Z}$, we conclude that $d = D_0$, and each of the inequalities above is actually an equality. Since $G_0(v) = \int_{r_0}^{R_0} e(r; v)\, dr = G_0(e^{iD_0\theta})$ and the integrands are pointwise bounded by (4.20), we conclude that equality must hold (almost everywhere) in (4.20), and therefore also in (4.19). The case of equality in the Cauchy–Schwarz inequality in the integrals over S_r implies that $(iv, \nabla v) \cdot \tau = \alpha(r)$ (independent of θ). Since the degree is independent of S_r, we have

$$2\pi \alpha(r) r = \int_0^{2\pi} (iv, \nabla v) \cdot \tau\, r\, d\theta = 2\pi D_0.$$

By the equality $|(iv, \nabla v) \cdot \tau| = |\nabla v|$ and since $|v| = 1$, we conclude that the normal derivative $(iv, \nabla v) \cdot n$ is equal to 0. Integrating the relation $(iv, \nabla v) \cdot \tau = D_0/r$ yields $v = e^{iD(\theta - \theta_0)}$ with $\theta_0 \in \mathbf{R}$ constant. □

Proof of Theorem 4.2: We first derive an upper bound on the energy of the minimizers. The splitting of the energy yields

$$E_\varepsilon(\eta_\varepsilon e^{iD_0\theta}) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon(e^{iD_0\theta}). \tag{4.21}$$

Since η_ε^2 converges to ρ_{TF}, $\lim_{\varepsilon \to 0} \mathcal{E}_\varepsilon(e^{iD_0\theta}) = g_0(D_0)$. For the minimizer u_ε, let $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$. We have $E_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon(v_\varepsilon)$. We thus get

$$\limsup_{\varepsilon \to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \leq g_0(D_0). \tag{4.22}$$

Now we want to prove that $\liminf_{\varepsilon\to 0}\mathcal{E}_\varepsilon(v_\varepsilon)\geq g_0(D_0)$. We are going to show some weak compactness of v_ε in the annulus to get the lower bound:

$$\begin{aligned}\left|\int_{\mathcal{D}}\eta_\varepsilon^2\mathbf{r}^\perp\cdot(iv,\nabla v)\right| &\leq \frac{1}{4}\int_{\mathcal{D}}\eta_\varepsilon^2|\nabla v|^2+\frac{\Omega^2}{2}\int_{\mathcal{D}}\eta_\varepsilon^2|x|^2|v|^2\\ &\leq \frac{1}{4}\int_{\mathcal{D}}\eta_\varepsilon^2|\nabla v|^2+C\Omega^2\left[\int_{\mathcal{D}}\eta_\varepsilon^2(|v|^2-1)+\int_{\mathcal{D}}\eta_\varepsilon^2\right]\\ &\leq \frac{1}{4}\int_{\mathcal{D}}\eta_\varepsilon^2|\nabla v|^2+\frac{1}{8\varepsilon^2}\int_{\mathcal{D}}\eta_\varepsilon^4(|v|^2-1)^2\\ &\quad +C\Omega^2\left[\int_{\mathcal{D}}|x|^4+\int_{\mathcal{D}}\rho_{\mathrm{TF}}+1\right].\end{aligned}$$

Therefore, this and (4.22) imply

$$g_0(D_0)+o(1)\geq \frac{1}{4}\int_{\mathcal{D}}\eta_\varepsilon^2|\nabla v_\varepsilon|^2+\frac{1}{8\varepsilon^2}\int_{\mathcal{D}}\eta_\varepsilon^4(|v_\varepsilon|^2-1)^2+O(\Omega^2).$$

As a consequence, there exists a constant C (depending on Ω but independent of ε) such that

$$\int_{\mathcal{D}}\left(\eta_\varepsilon^2|\nabla v|^2+\frac{1}{\varepsilon^2}\eta_\varepsilon^4(|v|^2-1)^2\right)\leq C. \tag{4.23}$$

In particular, along some subsequence we have $\eta_\varepsilon\nabla v_\varepsilon\rightharpoonup w_0$ weakly and $|v_\varepsilon|\to 1$ strongly in $L^2(\mathcal{D})$. Fix any $\delta>0$ and let $\mathcal{D}_\delta:=\{x\in\mathcal{D}:\ \operatorname{dist}(x,\partial\mathcal{D})>\delta\}$. Then

$$C\geq\int_{\mathcal{D}_\delta}\eta_\varepsilon^2|\nabla v_\varepsilon|^2\geq\frac{1}{2}\int_{\mathcal{D}_\delta}\rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2,$$

uniformly in ε. Hence, v_ε is bounded in $H^1(\mathcal{D}_\delta)$ for each $\delta>0$, and a subsequence converges weakly in $H^1_{\mathrm{loc}}(\mathcal{D}_\delta)$, strongly in $L^2(\mathcal{D}_\delta)$, and pointwise almost everywhere. By a diagonal argument we obtain a limiting function v_0, with $|v_0|=1$ and $v_\varepsilon\rightharpoonup v_0$ in $\mathcal{D}_\delta$ for each $\delta>0$. By lower semicontinuity, for each $\delta>0$,

$$\liminf_{\varepsilon\to 0}\int_{\mathcal{D}}\eta_\varepsilon^2|\nabla v_\varepsilon|^2\geq\liminf_{\varepsilon\to 0}\int_{\mathcal{D}_\delta}\eta_\varepsilon^2|\nabla v_\varepsilon|^2\geq\int_{\mathcal{D}_\delta}\rho_{\mathrm{TF}}|\nabla v_0|^2.$$

We let $\delta\to 0$ and obtain $v_0\in H^1_{\rho_{\mathrm{TF}}}(\mathcal{D})$. By the pointwise convergence, we may also identify the weak limit w_0 above: we have $\eta_\varepsilon\nabla v_\varepsilon\rightharpoonup\sqrt{\rho_{\mathrm{TF}}}\nabla v_0$ weakly in $L^2(\mathcal{D})$. We want to show that this convergence is in fact strong.

The rotation term also converges away from the boundary: by weak convergence of ∇v_ε, strong convergence of v_ε, and uniform convergence of η_ε^2 in $\mathcal{D}_\delta$, we have for each $\delta>0$,

$$\Omega\int_{\mathcal{D}_\delta}\eta_\varepsilon^2\mathbf{r}^\perp\cdot(iv_\varepsilon,\nabla v_\varepsilon)\to\Omega\int_{\mathcal{D}_\delta}\rho_{\mathrm{TF}}\mathbf{r}^\perp\cdot(iv_0,\nabla v_0). \tag{4.24}$$

In particular, $\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon; \mathcal{D}_\delta) \geq G_0(v_0; \mathcal{D}_\delta)$. Let $\mathcal{N}_\delta = \mathcal{D}\setminus\mathcal{D}_\delta$. By Lemma 4.3, (4.22), (4.24), and the above, we have that for any $\gamma > 0$, we may choose ε sufficiently small, so that

$$\gamma + G_0(v_0) \geq \gamma + G_0(e^{iD_0\theta}) \geq \mathcal{E}_\varepsilon(v_\varepsilon) \geq G_0(v_0; \mathcal{D}_\delta) + \mathcal{E}_\varepsilon(v_\varepsilon; \mathcal{N}_\delta) - \gamma, \quad (4.25)$$

and hence we may choose $\varepsilon > 0$ small enough so that

$$\mathcal{E}_\varepsilon(v_\varepsilon; \mathcal{N}_\delta) \leq 2\gamma + G_0(v_0; \mathcal{N}_\delta) \leq 3\gamma. \quad (4.26)$$

This allows us to estimate the rotation term in $\mathcal{N}_\delta$ in a similar way as before, for $\delta > 0$ sufficiently small:

$$\begin{aligned}
&|\Omega \int_{\mathcal{N}_\delta} \eta_\varepsilon \mathbf{r}^\perp \cdot (iv_\varepsilon, \nabla v_\varepsilon)| \\
&\leq \frac{1}{4}\int_{\mathcal{N}_\delta} \eta_\varepsilon^2 |\nabla v_\varepsilon|^2 + C\Omega^2 \left[\left(\int_{\mathcal{D}} \eta_\varepsilon^4 (|v_\varepsilon|^2 - 1)^2 \right)^{\frac{1}{2}} + (\max \rho_{\mathrm{TF}}) |\mathcal{N}_\delta| \right] \\
&\leq \mathcal{E}_\varepsilon(v_\varepsilon; \mathcal{N}_\delta) + \gamma \leq 4\gamma,
\end{aligned}$$

for ε, δ sufficiently small. Together with (4.24) we conclude that

$$\lim_{\varepsilon\to 0} \int_{\mathcal{D}} \eta_\varepsilon \mathbf{r}^\perp \cdot (iv_\varepsilon, \nabla v_\varepsilon) = \int_{\mathcal{D}} \rho_{\mathrm{TF}} \mathbf{r}^\perp \cdot (iv_0, \nabla v_0),$$

and from above,

$$\liminf_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) \geq G_0(v_0).$$

Therefore $\lim_{\varepsilon\to 0} \mathcal{E}_\varepsilon(v_\varepsilon) = G_0(v_0)$, so that $\lim \int_{\mathcal{D}} \eta_\varepsilon |\nabla v_\varepsilon|^2 = \int_{\mathcal{D}} \rho_{\mathrm{TF}} |\nabla v_0|^2$, and hence $\eta_\varepsilon \nabla v_\varepsilon \to \sqrt{\rho_{\mathrm{TF}}}\nabla v_0$ strongly in $L^2(\mathcal{D})$, i.e., $v_\varepsilon \to v_0$ strongly in $H^1_{\mathrm{loc}}(\mathcal{D})$, with v_0 a minimizer of G_0, that is, $v_0 = \alpha e^{iD_0\theta}$ with $|\alpha| = 1$.

The uniform convergence of $|v_\varepsilon| \to 1$ in $\mathcal{D}_\delta$ for any $\delta > 0$ follows from the same arguments as Step A.2 of the proof of Theorem 1 of [33], since the matching of the upper and lower bounds on $\mathcal{E}_\varepsilon(v_\varepsilon; \mathcal{D})$ implies

$$\frac{1}{\varepsilon^2}\int_{\mathcal{D}} \eta_\varepsilon^4 (|v|^2 - 1)^2 = o(1).$$

This completes the proof of Theorem 4.2. □

A recent paper of André, Bauman, and Phillips [21] deals with the case when $\rho_{\mathrm{TF}}(x)$ is allowed to vanish at isolated points. Their model originates from superconductivity. They show that when the applied field (which plays the role of our angular speed Ω) is large but *fixed* (independent of ε), minimizers have nonzero degree. The vortices are pinned to the zeros of ρ_{TF}, and none appear in the region where $\rho_{\mathrm{TF}} > 0$. In their result, it is important that $\sqrt{\rho}_{\mathrm{TF}}$ is in H^1, which is not the case here. In particular, the profile of the condensate is singular near the boundary, and contributes to a divergent term in the expansion of energy. We overcome this difficulty by a splitting of the energy to separate the contribution of the vortices from that of the singular boundary layer.

4.2.2 Circle of vortices

As in the model case $\rho_0 - r^2$, vortices become energetically favorable in the bulk $\mathcal{D}$ at a critical value of the rotation $\Omega^* = O(|\log\varepsilon|)$. In our model case, they are nucleated close to the origin, because it is the location of the maximum of the potential function ξ/ρ_{TF}, proportional to ρ_{TF}. We show that the same general principle holds in the annular case: vortices are nucleated close to the point of maximum of ξ/ρ_{TF}, which is now an inner circle. This is due to the fact that giant vortex exerts a repulsive force on free vortices in the interior of $\mathcal{D}$, which effectively balances the force of the rotation. Hence vortices lie on a specific circle.

We will identify $\mathcal{D}$ with the annulus $B_{R_0} \setminus \overline{B}_{R_1}$.

Theorem 4.4. *We assume a specific asymptotic form for the rotation* Ω*:*

$$\Omega = \omega_0|\log\varepsilon| + \omega_1 \log|\log\varepsilon|. \tag{4.27}$$

Let ρ *be such that* $(\log|\log\varepsilon|)^{-1/2} \ll \rho \ll 1$, $\mathcal{D}_\rho$ *be defined as*

$$\mathcal{D}_\rho = \{\mathbf{r} \in \mathcal{D},\ \operatorname{dist}(\mathbf{r}, \partial\mathcal{D}) > \rho_\varepsilon\}. \tag{4.28}$$

Then, there exist constants ω_0^*, ω_1^* *such that if* u_ε *is a sequence of minimizers of* E_ε *in* $H_0^1(\mathcal{D})$*, the following hold:*

(i) If either $\omega_0 < \omega_0^*$ *or both* $\omega_0 = \omega_0^*$ *and* $\omega_1 < -\omega_1^*$*, then for all* ε *sufficiently small,* $|u_\varepsilon|$ *does not vanish in* $\mathcal{D}_\rho$*.*

(ii) If $\omega_0 = \omega_0^*$ *and* $\omega_1 > 0$*, then for all* ε *sufficiently small, any vortex in* $\mathcal{D}_\rho$ *has positive degree and is localized on the circle* $\mathcal{C}$ *of radius* r_0 *characterized by the fact that* r_0 *minimizes* $\xi(r)/\rho_{\mathrm{TF}}(r)$ *in* (R_1, R_0)*, where*

$$\xi(r) := \int_r^{R_0} \rho_{\mathrm{TF}}(s)\left(s - \frac{1}{\Lambda_1 s}\right) ds, \quad \Lambda_1 = \int_{\mathcal{D}} \frac{\rho_{\mathrm{TF}}(s)}{s^2}. \tag{4.29}$$

For any circle C_r *with* $r > r_0$*,*

$$\deg\left(\frac{u}{|u|}, C_r\right) - \deg\left(\frac{u}{|u|}, C_\rho\right) > 0.$$

Note that given the definition of Λ_1 and the constraint on ρ_0 in (4.3), $\xi(R_1) = 0$. As in the previous chapter, there may exist vortices near the edges of $\mathcal{D}$, but the value of ρ_{TF} being very small near $\partial\mathcal{D}$, we have no way of controlling these outlying vortices.

As previously, we decouple the energy of $u_\varepsilon = \eta_\varepsilon v_\varepsilon$ into the energy of η_ε plus another term that we need to estimate. But we refine our splitting of energy to incorporate the effect of the giant vortex: we use the configuration $\eta_\varepsilon e^{iD_\varepsilon\theta}$ in fact as comparison function. Our asymptotic expansion of the energy leads to the appearance of a new potential function ξ defined in (4.29): $\xi(r) \geq 0$, $\xi(R_1) = \xi(R_0) = 0$, and

$$\max_{r\in[R_1,R_0]} \frac{\xi(r)}{\rho_{\mathrm{TF}}(r)} =: K_0 > 0, \tag{4.30}$$

is attained at r_0. We show that $\omega_0^* = 1/(2K_0)$ gives the desired critical value of rotation, in the sense that when $\omega_0 < \omega_0^*$, minimizers have no vortices inside $\mathcal{D}_\rho$. On the other hand, if $\omega_0 = \omega_0^*$ and $\omega_1 > 0$, vortices converge to the circle $\mathcal{C}$ of radius r_0. The proof follows the arguments in Section 3.3: the lower bound on the energy leads to the upper bound on the number of vortices, the positivity of the degrees, and their location on $\mathcal{C}$.

The rest of this section is devoted to the proof of Theorem 4.4.

Splitting the energy

We refine our splitting of energy to incorporate the effect of the giant vortex. We define v_ε through

$$u_\varepsilon = \eta_\varepsilon e^{iD_\varepsilon\theta} v_\varepsilon, \text{ with } D_\varepsilon = \left[\frac{\Omega}{\Lambda_1}\right], \tag{4.31}$$

where $[x]$ denotes the closest integer to x and Λ_1 is given by (4.14). Recall that $\Omega = O(|\log\varepsilon|)$. Since $e^{iD_\varepsilon\theta}$ is smooth and of modulus one in the annulus $\mathcal{D}$, it follows that v is well defined, and (4.15) and a direct calculation yield

$$\begin{aligned} E_\varepsilon(u_\varepsilon) &= E_\varepsilon(\eta_\varepsilon) + \mathcal{E}_\varepsilon(e^{iD_\varepsilon\theta} v_\varepsilon) \\ &= E_\varepsilon(\eta_\varepsilon) + \frac{1}{2}\Lambda_\varepsilon D_\varepsilon^2 - \Omega\, M_\varepsilon D_\varepsilon + \tilde{\mathcal{E}}_{\eta_\varepsilon}(v_\varepsilon), \end{aligned} \tag{4.32}$$

with

$$\Lambda_\varepsilon = \int_{\mathcal{D}} \frac{\eta_\varepsilon^2}{|x|^2} \text{ and } M_\varepsilon = \int_{\mathcal{D}} \eta_\varepsilon^2 |v_\varepsilon|^2, \tag{4.33}$$

$$\tilde{\mathcal{E}}_{\eta_\varepsilon}(v) = \int_{\mathcal{D}} \frac{\eta_\varepsilon^2}{2}|\nabla v|^2 - \eta_\varepsilon^2 \Omega \mathbf{X}\cdot(iv, \nabla v) + \frac{\eta_\varepsilon^4}{4\varepsilon^2}(|v|^2-1)^2, \tag{4.34}$$

where $\mathbf{X} = x^\perp - \frac{D_\varepsilon}{\Omega}\nabla\theta$.

Using $\eta_\varepsilon e^{iD_\varepsilon\theta}$ as a test function for an upper bound, we find that if u_ε is a minimizer, then $\tilde{\mathcal{E}}_{\eta_\varepsilon}(v_\varepsilon) \le 0$. Our aim is to compute a lower bound and thus locate the vortices. The rest of the proof follows the lines of Section 3.3.

Vortex balls

Our first step is to excise a thin neighborhood of the two edges where $\rho_{\mathrm{TF}}(r)$ vanishes. Let

$$\delta = \delta_\varepsilon = \frac{(\log|\log\varepsilon|)^{1/4}}{|\log\varepsilon|}, \tag{4.35}$$

and

$$\mathcal{D}_{\delta_\varepsilon} := \{\mathbf{r} \in \mathcal{D} : \ \mathrm{dist}(\mathbf{r}, \partial\mathcal{D}) > \delta_\varepsilon\}, \qquad \mathcal{N}_\varepsilon := \mathcal{D} \setminus \mathcal{D}_{\delta_\varepsilon}.$$

Then, by familiar arguments we have

$$\Omega \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 \mathbf{X} \cdot (iv, \nabla v) \le \frac{1}{2} \int_{\mathcal{N}_\varepsilon} \eta_\varepsilon^2 |\nabla v|^2 + \int_{\mathcal{N}_\varepsilon} \frac{\eta_\varepsilon^4}{8\varepsilon^2} (|v|^2 - 1)^2 + C\sqrt{\log|\log \varepsilon|}. \tag{4.36}$$

In particular, $\tilde{\mathcal{E}}_{\eta_\varepsilon}(v; \mathcal{N}_\varepsilon) \ge -C\sqrt{\log|\log \varepsilon|}$, and consequently,

$$\tilde{\mathcal{E}}_{\eta_\varepsilon}(v; \mathcal{D}_{\delta_\varepsilon}) \le C\sqrt{\log|\log \varepsilon|} \tag{4.37}$$

for any minimizer.

Note that by the same steps as in (3.36) above,

$$\Omega \int_{\mathcal{D}_{\delta_\varepsilon}} \eta_\varepsilon^2 \mathbf{X} \cdot (iv, \nabla v) \le \frac{1}{4} \int_{\mathcal{D}_{\delta_\varepsilon}} \eta_\varepsilon^2 |\nabla v|^2 + \frac{1}{8\varepsilon^2} \int_{\mathcal{D}_{\delta_\varepsilon}} \eta_\varepsilon^4 (|v|^2 - 1)^2 + C\Omega^2, \tag{4.38}$$

and hence from (4.37) we obtain the useful estimate

$$\int_{\mathcal{D}_{\delta_\varepsilon}} \left\{ \eta_\varepsilon^2 |\nabla v|^2 + \frac{\eta_\varepsilon^4}{\varepsilon^2} (|v|^2 - 1)^2 \right\} \le C\Omega^2 = O(|\log \varepsilon|^2), \tag{4.39}$$

with C independent of ε.

We now define

$$\tilde{\mathcal{E}}_\varepsilon(v) := \int_{\mathcal{D}_{\delta_\varepsilon}} \left\{ \frac{\rho_{\mathrm{TF}}}{2} |\nabla v|^2 - \Omega \rho_{\mathrm{TF}} \mathbf{X} \cdot (iv, \nabla v) + \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2} (|v|^2 - 1)^2 \right\}, \tag{4.40}$$

and using the estimates on $(\eta_\varepsilon^2 - \rho_{\mathrm{TF}})$, we conclude that

$$\tilde{\mathcal{E}}_\varepsilon(v) \le \tilde{\mathcal{E}}_{\eta_\varepsilon}(v; \mathcal{D}_{\delta_\varepsilon}) \left(1 + o(\varepsilon^{1/3} |\log \varepsilon|) \right) \le C\sqrt{\log|\log \varepsilon|}. \tag{4.41}$$

Moreover, the bounds (4.38), (4.39) also hold with ρ_{TF} replacing η_ε^2:

$$\int_{\mathcal{D}_{\delta_\varepsilon}} \rho_{\mathrm{TF}} |\nabla v|^2 + \frac{\rho_{\mathrm{TF}}^2}{\varepsilon^2} (|v|^2 - 1)^2, \ \left| \Omega \int_{\mathcal{D}_{\delta_\varepsilon}} \rho_{\mathrm{TF}} \mathbf{X} \cdot (iv, \nabla v) \right| \le C\Omega^2. \tag{4.42}$$

Now in $\mathcal{D}_{\delta_\varepsilon}$ we may isolate the vortices using the method of Sandier [133] and Sandier and Serfaty [135]. We have the following result:

Proposition 4.5. *For any $C > 0$ there exist positive constants ε_0, C_0 such that for any $\varepsilon < \varepsilon_0$, $\Omega \le C|\log \varepsilon|$, and any v with $\tilde{\mathcal{E}}_\varepsilon(v) \le C/|\log \varepsilon|$ there exists a finite collection $\{B_i = B(p_i, s_i)\}_{i=1,\dots,m}$ of disjoint balls such that*

$$\left\{x \in \mathcal{D}_{\delta_\varepsilon} \;:\; |v| < 1 - |\log \varepsilon|^{-5}\right\} \subset \bigcup_{i=1}^{m} B_i \;; \tag{4.43}$$

$$\sum_{i=1}^{m} s_i < |\log \varepsilon|^{-10}; \tag{4.44}$$

$$\deg_{\partial B_i}\left(\frac{v}{|v|}\right) := d_i \qquad \textit{for all } i; \tag{4.45}$$

$$\int_{B_i} \frac{\rho_{\mathrm{TF}}}{2} |(\nabla - i\Omega \mathbf{X})v|^2 \geq \pi \rho_{\mathrm{TF}}(p_i)|d_i|\,(|\log \varepsilon| - C_0 \log|\log \varepsilon|) \;\; \textit{for all } i. \tag{4.46}$$

Potential function ξ

For $\delta > 0$ and $\theta \in \mathbf{R}$, let us define

$$\xi_{\theta,\delta}(r) := \int_r^{R_0-\delta} \rho_{\mathrm{TF}}(s)\left(s - \left(\frac{1}{\Lambda_1} - \theta\right)\frac{1}{s}\right) ds. \tag{4.47}$$

Here we take $\delta = \delta_\varepsilon$ as in (4.35), and

$$\theta = \theta_\varepsilon = \frac{1}{\Lambda_1} - \frac{D_\varepsilon}{\Omega} = \left(\frac{\Omega}{\Lambda_1} - \left[\frac{\Omega}{\Lambda_1}\right]\right)\frac{1}{\Omega},$$

so that

$$|\theta_\varepsilon| \leq \frac{1}{\Omega} = O(|\log \varepsilon|^{-1}). \tag{4.48}$$

The reason why $\xi_{\theta_\varepsilon,\delta_\varepsilon}$ enters into our problem is that it is the primitive of the vector field $\rho_{\mathrm{TF}}(r)\mathbf{X}$ that vanishes at the outer edge of $\mathcal{D}_{\delta_\varepsilon}$:

$$\nabla^{\perp}\left(\xi_{\theta_\varepsilon,\delta_\varepsilon}(r)\right) = \rho_{\mathrm{TF}}(r)\,\mathbf{X}, \qquad \xi_{\theta,\delta}(R_0 - \delta) = 0, \tag{4.49}$$

for any $\delta_\varepsilon, \theta_\varepsilon$. It will be important later on to notice that

$$\xi'_{\theta,\delta}(R_1) = 0 = \xi'_{\theta,\delta}(R_0), \tag{4.50}$$

and when $\theta = 0 = \delta$,

$$\xi_{0,0}(R_1) = 0 = \xi_{0,0}(R_0), \qquad \xi_{0,0}(r) > 0 \text{ for all } r \in (R_1, R_0). \tag{4.51}$$

Lemma 4.6. *Let r_0 be the point where $\xi_{0,0}(r)/\rho_{\mathrm{TF}}(r)$ reaches its maximum and let $K_0 = \xi_{0,0}(r_0)/\rho_{\mathrm{TF}}(r_0)$. There exist constants $K_2, K_3 > 0$ such that*

$$\frac{|\xi_{\theta_\varepsilon,\delta_\varepsilon}(r)|}{\rho_{\mathrm{TF}}(r)} \leq K_0 + \frac{K_2}{|\log \varepsilon|} \tag{4.52}$$

whenever $r \in (R_1 + \delta_\varepsilon, R_0 - \delta_\varepsilon)$, and

$$\frac{|\xi_{\theta_\varepsilon,\delta_\varepsilon}(r)|}{\rho_{\mathrm{TF}}(r)} \leq K_0 - \frac{K_3}{\sqrt{|\log \varepsilon|}} \tag{4.53}$$

when $|r - r_0| \geq |\log \varepsilon|^{-1/2M}$ for some $M \geq 1$.

Proof: We have

$$|\xi_{\theta_\varepsilon,\delta_\varepsilon}(r) - \xi_{0,0}(r)| \le C(|\theta_\varepsilon| + {\delta_\varepsilon}^2), \tag{4.54}$$

with a constant C independent of ε. There exist constants α and a_0 such that for any $\gamma > 0$ sufficiently small,

$$\rho_{\mathrm{TF}}(r) \ge \begin{cases} \alpha(r - R_1), & \text{if } R_1 < r < R_1 + \gamma, \\ a_0, & \text{if } R_1 + \gamma \le r \le R_0 - \gamma, \\ \alpha(R_0 - r), & \text{if } R_0 - \gamma < r < R_0. \end{cases}$$

When $R_1 + \delta_\varepsilon < r < R_1 + \gamma$, we calculate

$$|\xi_{\theta_\varepsilon,\delta_\varepsilon}(r)| \le |\xi_{0,0}(r)| + C(|\theta_\varepsilon| + {\delta_\varepsilon}^2) \le C(r - R_1)^2 + C(|\theta_\varepsilon| + {\delta_\varepsilon}^2),$$

using (4.49), (4.50), and (4.51). Therefore,

$$\frac{|\xi_{\theta_\varepsilon,\delta_\varepsilon}(r)|}{\rho_{\mathrm{TF}}(r)} \le \frac{C}{\alpha}\gamma + \frac{C}{\alpha}\left(\frac{|\theta_\varepsilon| + {\delta_\varepsilon}^2}{\delta_\varepsilon}\right) \le C\gamma + O([\log|\log\varepsilon|]^{-1/4}) < \frac{1}{2}K_0, \tag{4.55}$$

by fixing a value of γ sufficiently small. An analogous estimate holds on the interval $[R_0 - \gamma, R_0 - \delta_\varepsilon]$.

It remains to consider the larger interval $R_1 + \gamma \le r \le R_0 - \gamma$. Since

$$\left|\frac{\xi_{\theta_\varepsilon,\delta_\varepsilon}(r)}{\rho_{\mathrm{TF}}(r)} - \frac{\xi_{0,0}(r)}{\rho_{\mathrm{TF}}(r)}\right| \le \frac{C}{a_0}(|\theta_\varepsilon| + {\delta_\varepsilon}^2) \le C|\log\varepsilon|^{-1},$$

the conclusion follows from the definition of r_0. □

A lower-bound expansion

We define $\mathcal{C}$ to be the circle of radius r_0.

The proof is based on a detailed lower-bound expansion of the energy in terms of the location and degrees of the vortex balls (B_i) constructed in Proposition 4.5. First, we consider the energy in the balls themselves. We have

$$\begin{aligned}\int_{B_i} \frac{\rho_{\mathrm{TF}}}{2}\left(|\nabla v|^2 - 2\Omega\mathbf{X}\cdot(iv, \nabla v)\right) &= \int_{B_i} \frac{\rho_{\mathrm{TF}}}{2}\left(|(\nabla - i\Omega\mathbf{X})v|^2 - \Omega^2|X|^2|v|^2\right) \\ &\ge \pi\rho_{\mathrm{TF}}(p_i)|d_i|(|\log\varepsilon| - C_0\log|\log\varepsilon|) - o(1),\end{aligned} \tag{4.56}$$

from Proposition 4.5, where we have estimated the extra term using (4.44). We may also evaluate the cross-term in the region $\mathcal{D}_\varepsilon \setminus \cup B_i$ in terms of the potential functions $\xi_{\theta_\varepsilon,\delta_\varepsilon}$ introduced in the previous paragraph. First note that by slightly modifying our choice of δ_ε we may be sure that no vortex ball intersects the inner or outer boundary $\partial B_{R_1+\delta_\varepsilon}(0)$, $\partial B_{R_0-\delta_\varepsilon}(0)$ of the annulus $\mathcal{D}_{\delta_\varepsilon}$. Indeed, if this is not the case by (4.44) we may find a constant $k_\varepsilon \in [1, 2)$ such that replacing ${\delta_\varepsilon}' = k_\varepsilon\delta_\varepsilon$ we may avoid vortex balls intersecting the boundary.

Lemma 4.7. *Let* $d_0 = \deg(v/|v|; \partial B_{R_1+\delta_\varepsilon})$. *Then,*

$$\Omega \int_{\mathcal{D}_{\delta_\varepsilon} \setminus \cup B_i} \rho_{\mathrm{TF}}(x)\, \mathbf{X} \cdot (iv, \nabla v) = -2\pi d_0\, \xi_{\theta_\varepsilon, \delta_\varepsilon}(R_1 + \delta_\varepsilon)\, \Omega + \sum_i 2\pi d_i\, \xi_{\theta_\varepsilon, \delta_\varepsilon}(|p_i|)\, \Omega + o(1). \quad (4.57)$$

The proof is similar to that of Lemma 3.11. We do not repeat it. The only difference lies in the existence of d_0, which is estimated similarly but with the opposite sign.

Putting (4.56) and (4.57) together we obtain the lower bound

$$\begin{aligned} &O(\sqrt{\log|\log \varepsilon|}) \geq \tilde{\mathcal{E}}_\varepsilon(v) \\ &\quad \geq \pi \sum \rho_{\mathrm{TF}}(p_i)|d_i|\,(|\log \varepsilon| - C_0 \log|\log \varepsilon|) - 2\pi \sum d_i \xi_{\theta_\varepsilon, \delta_\varepsilon}(|p_i|)\, \Omega \\ &\quad -2\pi d_0 \xi_{\theta_\varepsilon, \delta_\varepsilon}(R_1 + \delta_\varepsilon)\, \Omega + \frac{1}{2} \int_{\mathcal{D}_{\delta_\varepsilon} \setminus \cup B_i} \rho_{\mathrm{TF}}\, |\nabla v|^2 + o(1). \end{aligned} \quad (4.58)$$

The behavior of $\rho_{\mathrm{TF}}(r)$ and $\xi_{0,0}(r)$ allow us to choose $\gamma > 0$ (independent of ε) such that

$$R_1 + \gamma < r_0 < R_0 - \gamma,$$

$$\rho_{\mathrm{TF}}(r) \geq a_0 := \min\{\rho_{\mathrm{TF}}(R_1 + \gamma), \rho_{\mathrm{TF}}(R_0 - \gamma)\} \quad \text{for all } r \in [R_1 + \gamma, R_0 - \gamma],$$

and

$$\frac{\xi_{0,0}(r)}{\rho_{\mathrm{TF}}(r)} < \frac{K_0}{3} \qquad \text{for all } r \in (R_1, R_1 + \gamma) \cup (R_0 - \gamma, R_0). \quad (4.59)$$

Let

$$\begin{aligned} Z_\gamma &:= \{i : \ \delta_\varepsilon < \mathrm{dist}(p_i, \partial\mathcal{D}) \leq \gamma, \\ Z_* &:= \{i : \mathrm{dist}(p_i, \mathcal{C}) < |\log \varepsilon|^{-1/2M} \text{ and } d_i \geq 0\}, \\ Z_- &:= \{i : \mathrm{dist}(p_i, \mathcal{C}) < |\log \varepsilon|^{-1/2M} \text{ and } d_i \leq 0\}, \\ Z_0 &:= (Z_* \cup Z_- \cup Z_\gamma)^C, \end{aligned}$$

and set

$$N_x := \sum_{Z_x} \rho_{\mathrm{TF}}(p_i)|d_i|, \quad x = \gamma, *, -, 0; \quad \hat{N} = \sum_i \rho_{\mathrm{TF}}(p_i)|d_i|.$$

In fact, $\hat{N} = N_\gamma + N_* + N_- + N_0$.

For vortices p_i with $i \in Z_\gamma$, we use (4.55) and (4.59) to derive

$$\frac{|\xi_{\theta_\varepsilon, \delta_\varepsilon}(|p_i|)|}{\rho_{\mathrm{TF}}(p_i)} < \frac{K_0}{2}. \quad (4.60)$$

Lemma 4.6 and (4.58) then yield

$$
\begin{aligned}
C\sqrt{\log|\log\varepsilon|} \geq\ & \pi\hat{N}\,(|\log\varepsilon| - C_0\log|\log\varepsilon|) \\
& -2\pi K_0 N_* \left[1 + O(|\log\varepsilon|^{-1})\right](\omega_0|\log\varepsilon| + \omega_1\log|\log\varepsilon|) \\
& +2\pi K_0 N_- \left[1 + O(|\log\varepsilon|^{-1})\right](\omega_0|\log\varepsilon| + \omega_1\log|\log\varepsilon|) \\
& -2\pi K_0 N_\gamma \left[\frac{1}{2} + O(|\log\varepsilon|^{-1})\right](\omega_0|\log\varepsilon| + \omega_1\log|\log\varepsilon|) \\
& -2\pi K_0 N_0 \left[1 - \frac{K_4}{\sqrt{|\log\varepsilon|}}\right](\omega_0|\log\varepsilon| + \omega_1\log|\log\varepsilon|) \\
& -2\pi d_0 \xi_{\theta_\varepsilon,\delta_\varepsilon}(R_1+\delta_\varepsilon)\Omega + \int_{\mathcal{D}_{\delta_\varepsilon}\setminus\cup B_i}\frac{\rho_{\mathrm{TF}}}{2}|\nabla v|^2 + o(1). \qquad (4.61)
\end{aligned}
$$

One difficulty in dealing with this lower bound expansion is the boundary term at $R_1+\delta_\varepsilon$, since we have no a priori bound on the degree d_0 of the inner edge of the annulus. We must consider two cases separately.

Case I: $|d_0| \leq 2\sum|d_i|$.

Recalling (4.50), (4.51), and the behavior of $\rho_{\mathrm{TF}}(r)$ near $r = R_1$, we have

$$
0 < \frac{\xi_{0,0}(R_1+\delta_\varepsilon)}{\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon)} \leq C\delta_\varepsilon.
$$

With (4.54) we obtain

$$
\begin{aligned}
|\xi_{\theta_\varepsilon,\delta_\varepsilon}(R_1+\delta_\varepsilon)| &\leq C\delta_\varepsilon\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon) + C(|\theta_\varepsilon| + {\delta_\varepsilon}^2) \\
&\leq C\left(\delta_\varepsilon + \frac{|\theta_\varepsilon|}{\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon)}\right)\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon) \\
&\leq \frac{C}{[\log|\log\varepsilon|]^{1/4}}\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon).
\end{aligned}
$$

Hence,

$$
\left|d_0\,\Omega\,\xi_{\theta_\varepsilon,\delta_\varepsilon}(R_1+\delta_\varepsilon)\right| \leq \frac{C}{[\log|\log\varepsilon|]^{1/4}}\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon)|\log\varepsilon|\sum|d_i|. \qquad (4.62)
$$

Hence

$$
\begin{aligned}
2\pi|d_0\,\Omega\,\xi_{\theta_\varepsilon,\delta_\varepsilon}(R_1+\delta_\varepsilon)| &\leq \frac{C}{[\log|\log\varepsilon|]^{1/4}}\rho_{\mathrm{TF}}(R_1+\delta_\varepsilon)|\log\varepsilon|\sum_{Z_\gamma\cup Z_\gamma^C}|d_i| \\
&\leq \frac{C|\log\varepsilon|}{[\log|\log\varepsilon|]^{1/4}}\sum_{Z_\gamma}\rho_{\mathrm{TF}}(p_i)|d_i| + \frac{C\delta_\varepsilon}{[\log|\log\varepsilon|]^{1/4}}\sum_{Z_\gamma^C}\frac{\rho_{\mathrm{TF}}(p_i)}{a_0}|d_i| \\
&\leq \frac{\pi}{8}|\log\varepsilon|\sum_{Z_\gamma}\rho_{\mathrm{TF}}(p_i)|d_i| + \frac{C}{|\log\varepsilon|}\sum_{Z_\gamma^C}\rho_{\mathrm{TF}}(p_i)|d_i|.
\end{aligned}
$$

We now substitute back into the lower bound for the energy (4.61):

$$
\begin{aligned}
C\sqrt{\log|\log\varepsilon|} &\geq \pi(N_* + N_0)|\log\varepsilon|(1 - 2K_0\omega_0)\\
&+\pi N_-|\log\varepsilon|(1 + 2K_0\omega_0)(1 + o(1)) + \pi N_\gamma|\log\varepsilon|\left(\frac{7}{8} - K_0\omega_0\right)(1 + o(1))\\
&-\pi N_*(C_0 + 2K_0\omega_1)\log|\log\varepsilon| + CN_0\sqrt{|\log\varepsilon|} + \int_{\mathcal{D}_{\delta_\varepsilon}\setminus\cup B_i}\frac{\rho_{\mathrm{TF}}}{2}|\nabla v|^2. \qquad (4.63)
\end{aligned}
$$

In this step we have used Lemma 4.6, (4.60), and the choice of the angular speed (4.27).

Our first step is to conclude that there are no vortices in the bulk when the speed is too small. Let $\omega_0^* = \frac{1}{2K_0}$. From (4.63), we derive

$$
\omega_0 < \omega_0^*, \text{ and then } \hat{N} = \sum\rho_{\mathrm{TF}}(p_i)|d_i| \leq C\frac{\sqrt{\log|\log\varepsilon|}}{|\log\varepsilon|}.
$$

Suppose $\omega_0 = \omega_0^*$ and

$$
\omega_1 < -\frac{C_0}{2K_0}.
$$

Then each term on the right-hand side of (4.63) is nonnegative, and we conclude that

$$
\hat{N} = \sum\rho_{\mathrm{TF}}(p_i)|d_i| \leq C[\log|\log\varepsilon|]^{-1/2}.
$$

Because of the weight ρ_{TF} we cannot conclude that the total degree of vortices in $\mathcal{D}_{\delta_\varepsilon}$ is zero, but we can make that conclusion if we restrict our attention to a smaller domain. Let $\rho = \rho_\varepsilon \gg (\log|\log\varepsilon|)^{-1/2}$. Then

$$
\rho_\varepsilon \sum_{\mathrm{dist}(p_i,\partial\mathcal{D})>\rho_\varepsilon}|d_i| \leq \hat{N} \leq C[\log|\log\varepsilon|]^{-1/2},
$$

which implies

$$
\sum_{\mathrm{dist}(p_i,\partial\mathcal{D})>\rho_\varepsilon}|d_i| = o(1),
$$

that is, there are no vortices at any distance larger than $[\log|\log\varepsilon|]^{-1/2}$ from $\partial\mathcal{D}$ when the rotation is slower than this critical value.

When the angular speed is larger,

$$
\omega_1 > -\frac{C_0}{2K_0}, \qquad (4.64)
$$

we rearrange the terms in (4.63) to arrive at

$$
(N_0 + N_- + N_\gamma)\sqrt{|\log\varepsilon|} \leq CN_*\log|\log\varepsilon| + C\sqrt{\log|\log\varepsilon|}, \qquad (4.65)
$$

with C independent of ε. We are going to bound N_* and therefore infer that the essential contribution to the weighted sum of vortices in the bulk is due to positive-degree vortices concentrating at the minimal set $\mathcal{C}$. To complete the argument we must use the remaining term in the energy. Define $I_\varepsilon := (R_1 + \delta_\varepsilon, R_0 - \delta_\varepsilon)$. By Proposition 4.5 the set

$$J_\varepsilon := \left\{ r \in I_\varepsilon : \partial B_r(0) \cap \bigcup B_{s_i}(p_i) = \emptyset \right\}$$

is a finite union of intervals whose complement $|I_\varepsilon \setminus J_\varepsilon| < |\log \varepsilon|^{-12}$ has very small measure. For each $r \in J_\varepsilon$, $|v| \geq 1 - |\log \varepsilon|^{-4}$, and hence we may define

$$D(r) := \deg \left(\frac{v}{|v|}, \partial B_r(0) \right).$$

Let $r_1 := R_1 + \gamma$, $r_2 := R_0 - \gamma$, and fix any t_1, t_2 with

$$r_1 < t_1 < r_0 < t_2 < r_2.$$

Note that r_1, r_2, t_1, t_2 are all independent of ε. In (r_1, r_2), we recall that $\rho_{\mathrm{TF}}(r) \geq a_0 > 0$.

On the one hand,

$$|D(t_1) - D(r_1)| = \left| \sum_{t_1 < |p_i| < r_1} d_i \right| \leq \sum_{t_1 < |p_i| < r_1} \frac{\rho_{\mathrm{TF}}(p_i)}{a_0} |d_i| \leq \frac{N_0}{a_0} \leq o(1) N_*.$$

In the same way we show that $|D(r_2) - D(t_2)| \leq o(1) N_*$. Finally,

$$\begin{aligned} |D(t_2) - D(t_1)| &= \left| \sum_{t_1 < |p_i| < t_2} d_i \right| \geq \sum_{\substack{t_1 < |p_i| < t_2 \\ d_i \geq 0}} d_i - \sum_{\substack{t_1 < |p_i| < t_2 \\ d_i < 0}} d_i \\ &\geq \frac{1}{\bar{a}} N_* - C(N_- + N_0) \geq \frac{1}{\bar{a}} N_* (1 - o(1)). \end{aligned}$$

In particular, it follows that

$$\min\{|D(t_1)|, |D(t_2)|\} \geq \frac{1}{2\bar{a}} N_*. \tag{4.66}$$

Suppose that $|D(t_1)| \geq \frac{1}{2\bar{a}} N_*$. Then we have $|D(r)| \geq \frac{1}{4\bar{a}} N_*$ for every $r \in [r_1, t_1]$. Writing $v = |v| e^{i\phi}$ (for $|x| = r \in J_\varepsilon$) we estimate the remaining term in the energy as follows, using that in J_ε, $|v| \geq 1 - |\log \varepsilon|^{-4}$:

$$\begin{aligned} \int_{\mathcal{D}_{\delta_\varepsilon} \setminus \cup B_i} \frac{1}{2} |\nabla v|^2 &\geq \int_{J_\varepsilon} \int_0^{2\pi} \frac{\rho_{\mathrm{TF}}(r)}{2} |v|^2 |\nabla \phi|^2 \, r \, d\theta \, dr \\ &\geq \int_{J_\varepsilon} \int_0^{2\pi} \frac{\rho_{\mathrm{TF}}(r)}{2} |\nabla \phi|^2 \, r \, d\theta \, dr (1 + o(1)) \\ &\geq \pi \int_{J_\varepsilon} \frac{\rho_{\mathrm{TF}}(r)}{r} (D(r))^2 (1 + o(1)) \\ &\geq C \, N_*^2 (1 + o(1)). \end{aligned} \tag{4.67}$$

Returning to the estimate (4.63) we see that

$$C\sqrt{\log|\log\varepsilon|} \geq C_1 N_*^2 - C_2 N_* \log|\log\varepsilon| + o(1),$$

with constants C_1, C_2 independent of ε. We conclude that

$$N_* \leq C\log|\log\varepsilon|. \tag{4.68}$$

With (4.65) and (4.68) we have

$$\max\{N_-, N_0, N_\gamma\} \leq C\frac{(\log|\log\varepsilon|)^2}{|\log\varepsilon|^{1/2}}.$$

As before, we need to further restrict the domain in order to come to a conclusion as to the total degree in the annulus. Take any ρ_ε with

$$\rho_\varepsilon \gg \frac{(\log|\log\varepsilon|)^2}{|\log\varepsilon|^{1/2}},$$

and then

$$\sum_{\substack{\operatorname{dist}(p_i,\mathcal{C})>|\log\ \varepsilon|^{-1/2M}\\ \operatorname{dist}(p_i,\partial\mathcal{D})>\rho_\varepsilon}} |d_i| + \sum_{\substack{d_i<0\\ \operatorname{dist}(p_i,\partial\mathcal{D})>\rho_\varepsilon}} |d_i| \leq C\frac{(\log|\log\varepsilon|)^2}{|\log\varepsilon|^{1/2}\rho_{\mathrm{TF}}(\rho_\varepsilon)} \to 0.$$

Since the left-hand side is now an integer, it must be exactly zero for ε sufficiently small.

Finally, we consider the degree of the neighborhood of the edge of the annulus. From the previous paragraph we observe that $D(r) \equiv D_1$ is *constant* in the interval $r \in [R_1 + \rho_\varepsilon, t_1]$. We return to the lower bound (4.63) to obtain

$$\begin{aligned} C\sqrt{\log|\log\varepsilon|} + CN_*\log|\log\varepsilon| &\geq \int_{\mathcal{D}_{\delta_\varepsilon}\setminus\cup B_i} \frac{a}{2}|\nabla v|^2 \\ &\geq \pi\left|\int_{r_1}^{s_1} \frac{a}{r}(D(r))^2\,dr\right| + o(1) \\ &\geq CD_1^2. \end{aligned}$$

In particular, given the bound (4.68) we have

$$\left|\deg\left(\frac{v}{|v|}, \partial B_{R_1+\rho_\varepsilon}(0)\right)\right| = |D_1| \leq C\log|\log\varepsilon|.$$

Note that this confirms that we have made a good choice of the degree D_ε of the giant vortex, since for the original wave function u we have

$$\deg\left(\frac{u}{|u|}, \partial B_{R_1+\rho_\varepsilon}(0)\right) = D_\varepsilon + O\left(\log|\log\varepsilon|\right).$$

Let us check that for $\omega_1 > 0$, there is at least one vortex. We construct a test function of the form $u = \eta_\varepsilon e^{iD_\varepsilon\theta} v_{p_0}$, where v_{p_0} has a vortex on $\mathcal{C}$: $v_{p_0}(x) = f_\varepsilon(|x - p_0|)\frac{x-p_0}{|x-p_0|}$, where $|p_0| \in \mathcal{C}$, $f_\varepsilon(0) = 0$, and $f_\varepsilon(\hat{R}) = 1$. If we fix $\hat{R} > 0$ such that $\mathcal{D} \supset B_{\hat{R}}(p_0)$ we then have

$$
\begin{aligned}
\mathcal{E}(v_{p_0}) \ &\leq \int_{B_{\hat{R}}(p_0)} \left(\frac{\rho_{\mathrm{TF}}}{2} |\nabla f_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(f_\varepsilon^2 - 1)^2 + \frac{\rho_{\mathrm{TF}}}{r^2} f_\varepsilon^2 \right) \\
&\quad -2\pi\,\Omega\,\xi_{\theta_\varepsilon,\delta_\varepsilon}(|p_0|) + o(1) \leq \pi\rho_{\mathrm{TF}}(r_0)|\log\varepsilon| \\
&\quad -2\pi \left(\frac{1}{2K_0}|\log\varepsilon| + \omega_1 \log|\log\varepsilon| \right) \left(K_0\rho_{\mathrm{TF}}(r_0) + O(|\log\varepsilon|^{-1}) \right) + C \\
&\leq -2\pi K_0 \rho_{\mathrm{TF}}(r_0)\omega_1 \log|\log\varepsilon| + C.
\end{aligned}
$$

We now return to our lower bound from (4.63). We now know that N_0, N_-, N_γ are $o(1)$, and therefore with this improved upper bound we obtain

$$
-\pi N_*(C_0 + 2K_0\omega_1) \leq -2\pi K_0 \rho_{\mathrm{TF}}(r_0)\omega_1 + o(1).
$$

This can be rewritten in the form of a lower bound for N_*,

$$
N_* \geq \frac{2K_0\rho_{\mathrm{TF}}(r_0)\omega_1}{C_0 + 2K_0\omega_1} + o(1),
$$

and hence for $\omega_1 > 0$, we must have at least one vortex. This concludes the analysis for Case I.

Case II: $|d_0| > 2\sum |d_i|$**.**

Let $D(r)$, J_ε be as in the previous part, so

$$
|D(r)| = \left| d_0 + \sum_{|p_i| \leq r} d_i \right| \geq \frac{1}{2}|d_0| \qquad \text{for all } r \in J_\varepsilon.
$$

We then estimate as before,

$$
\begin{aligned}
\int_{\mathcal{D}_{\delta_\varepsilon} \setminus \cup B_i} \frac{\rho_{\mathrm{TF}}}{2}|\nabla v|^2 &\geq \int_{J_\varepsilon} \int_0^{2\pi} \frac{\rho_{\mathrm{TF}}}{2}|v|^2|\nabla\phi|^2 \\
&\geq \pi \int_{J_\varepsilon} \frac{\rho_{\mathrm{TF}}}{r}(D(r))^2\,dr + o(1) \\
&\geq \frac{\pi}{2} d_0^2 \int_{J_\varepsilon} \frac{\rho_{\mathrm{TF}}}{r}\,dr = C_1 d_0^2.
\end{aligned} \tag{4.69}
$$

On the other hand, in Case II, using the estimate

$$
\left| 2\pi d_0\,\Omega\,\xi_{\theta_\varepsilon,\delta_\varepsilon}(R_1 + \delta_\varepsilon) \right| \leq \frac{C}{(\log|\log\varepsilon|)^{1/4}} d_0\delta_\varepsilon|\log\varepsilon| \leq C_2|d_0|
$$

for $\xi_{\theta_\varepsilon,\delta_\varepsilon}$, from (4.58) we get

$$
\begin{aligned}
C\sqrt{\log|\log\varepsilon|} \ge \tilde{\mathcal{E}}_\varepsilon(v) \\
&\ge C_1 d_0^2 - C_2 d_0 + \pi \sum \rho_{\mathrm{TF}}(p_i)|d_i|\,(|\log\varepsilon| - C_0\log|\log\varepsilon|) \\
&\quad -2\pi \sum d_i \xi_{\theta_\varepsilon,\delta_\varepsilon}(|p_i|)\,\Omega + o(1).
\end{aligned}
$$

We may now repeat the same steps as in Case I (although we no longer need to distinguish Z_γ, N_γ) to derive

$$C\sqrt{\log|\log\varepsilon|} \ge C_1 d_0^2 - C_2|d_0| + C_3(N_0 + N_-)\sqrt{|\log\varepsilon|} - CN_*\log|\log\varepsilon|,$$

and hence

$$(N_- + N_0) \le C\sqrt{\frac{\log|\log\varepsilon|}{|\log\varepsilon|}} + C|d_0|\frac{\log|\log\varepsilon|}{\sqrt{|\log\varepsilon|}}.$$

This leads again to

$$C_1 d_0^2 - C|d_0|\log|\log\varepsilon| \le C\sqrt{\log|\log\varepsilon|}$$

and thus the same conclusions as in Case I. This completes the proof of Theorem 4.4. □

4.3 Open questions

4.3.1 Circle of vortices

Open Problem 4.1 *Assume that $\Omega = \omega_0^*|\log\varepsilon| + \omega_1$, and prove that according to the value of ω_1 in some interval $(\omega_1^n, \omega_1^{n+1})$, the minimizer has n vortices located on a specific circle of radius r_0.*

The difficulty relies on the improvement of (4.67) to $N_*^2\log|\log\varepsilon|$, which provides that N_* is bounded. Then the refined structure of vortices can be analyzed as in Section 3.4. A totally new feature, as mentioned after Open Problem 3.3, is that when there are n vortices in the system, they are no longer at distance $1/\sqrt{|\log\varepsilon|}$, but at distance of order 1. The logarithm involved in the renormalized energy w should be replaced by something else.

4.3.2 Giant vortex or isolated vortices

If one replaces the problem in $\mathcal{D} = B_{R_0} \setminus \overline{B}_{R_1}$ by the problem in B_{R_0} but with the same geometry for the set $\{\rho_{\mathrm{TF}} > 0\}$, then the results of Theorem 4.2 still hold. It is an open question to prove that the minimizer has a giant vortex of degree d at the origin or d vortices of degree 1 in the region B_{R_1} where its modulus is exponentially small.

5

High-Velocity and Quantum Hall Regime

When the velocity gets large, the size of the condensate and the number of vortices increase: a dense lattice is observed [1, 47, 58, 141], referred to as an Abrikosov lattice due to the analogy with superconductors. The description of the vortex lattice at high rotational velocity has been the focus of very recent papers in the condensed-matter physics community, starting with the seminal paper of Ho [79] and very recently by [64, 27, 49, 154, 147]. Our aim is to provide mathematical insight into the lattice pattern.

The mathematical interest of such a system can be related to homogeneous media, since there are two scales emerging: the size of vortices (of order 1) and the size of the condensate (much larger). In this regime, vortices have approximately the same size as their mutual distance, which is very different from the lower rotation regime. Hence different mathematical tools need to be introduced.

This chapter is dedicated to the study of minimizers $\psi \in L^2(\mathbf{R}^2, \mathbf{C})$ of

$$E_{\mathrm{LLL}}(\psi) = \int_{\mathbf{R}^2} \frac{(1-\Omega^2)}{2} r^2 |\psi|^2 + \frac{Na}{2}|\psi|^4 \text{ under } \int_{\mathbf{R}^2} |\psi|^2 = 1, \tag{5.1}$$

where $\psi e^{\Omega|z|^2/2}$ is a holomorphic function. Here Ω is the rotational velocity and tends to 1; N, a are given parameters; and $\mathbf{r} = (x, y)$ or equivalently $z = x + iy$. We will identify complex numbers and vectors in $\mathbf{R}^2$, and in particular dz will denote the two-dimensional Lebesgue measure $dx\,dy$. We will construct a test function (which we believe is close to the minimizer) and this will provide an upper bound for the energy. The lower bound and Γ convergence properties are still open.

We will first explain the reduction from the Gross–Pitaevskii energy to this problem, then present our main results, proofs, and open questions. The results rely on the works [5, 7, 8].

5.1 Introduction

5.1.1 Lowest Landau level

Let us first present how the Gross–Pitaevskii energy (1.17) can be reduced to (5.1). For fixed Ω, if the trapping frequencies along the x and y directions are much smaller than along the z direction (that is, β in (1.17) is large), it has been proved [98] that the wave function decouples into $\psi(x, y)\xi(z)$, where ξ is a Gaussian in the z direction and ψ minimizes the reduced two-dimensional Gross–Pitaevskii energy. In the fast-rotation regime, we will see below that the effective trapping frequencies in the x and y directions, $\sqrt{1-\Omega^2}$ and $\alpha\sqrt{1-\Omega^2}$, are much smaller, as Ω tends to 1, than the frequency in the z direction, which is fixed. Thus one can expect the same type of reduction into $\psi(x, y)\xi(z)$, where ξ is a fixed Gaussian in the z direction and ψ minimizes the reduced two-dimensional energy depending on Ω. The rigorous proof of the reduction to a two-dimensional problem is still open, but a 2D description is expected to be satisfactory. This is what we will use.

The energy taking into account the 2D reduction can be written (we set $\alpha = 1$ for simplicity)

$$E(\psi) = \int_{\mathbf{R}^2} \frac{1}{2}|\nabla\psi - i\mathbf{\Omega} \times \mathbf{r}\psi|^2 + \frac{1}{2}(1-\Omega^2)r^2|\psi|^2 + \frac{1}{2}Na|\psi|^4, \tag{5.2}$$

under $\int_{\mathbf{R}^2} |\psi|^2 = 1$. The rescaled rotational velocity is along the z axis: $\mathbf{\Omega} = \Omega\mathbf{e}_z$. With respect to (1.17), we have added and subtracted $\Omega^2 r^2|\psi|^2/2$ to get a complete square in the first term. In order for the trapping potential to remain stronger than the rotating force, we need to have $\Omega < 1$, so that the energy is bounded below. The minimization is performed in $\mathbf{R}^2$ and not just in a bounded domain, because the size of the condensate increases as Ω approaches 1.

This regime displays a strong analogy with quantum Hall physics: the first term in the energy is identical to the energy of a particle placed in a uniform magnetic field $2\mathbf{\Omega}$. The minimizers for

$$\int_{\mathbf{R}^2} \frac{1}{2}|\nabla\psi - i\mathbf{\Omega} \times \mathbf{r}\psi|^2 \text{ under } \int_{\mathbf{R}^2} |\psi|^2 = 1 \tag{5.3}$$

are well known [104] through the study of the eigenvalues of the operator $-(\nabla - i\mathbf{\Omega} \times \mathbf{r})^2$. The minimum is Ω and the corresponding eigenspace is of infinite dimension and called the lowest Landau level (LLL). This can be studied using a change of gauge and a Fourier transform in one direction. The other eigenvalues are $(2k+1)\Omega$, $k \in \mathbf{N}$. A basis of the first eigenspace is given by

$$\psi(x, y) = P(z)e^{-\Omega|z|^2/2} \quad \text{with } z = x + iy, \tag{5.4}$$

where P varies in a basis of polynomials. The closure of this space in $L^2(\mathbf{R}^2)$ is made up of functions of the type (5.4) where P varies in the space of holomorphic functions. In this framework, vortices are the zeros of the polynomial or holomorphic function and are thus easy to identify.

We will see that as Ω approaches 1, the second and third terms in the energy (5.2) produce a contribution of order $\sqrt{1-\Omega}$, which is much smaller than the gap 2Ω between two eigenvalues. Thus, it is natural, as a first step, to restrict to the eigenfunctions of the first eigenvalue and find the minimizer of the energy in this reduced infinite-dimensional space, but we are not able to provide a full rigorous justification of this restriction. When ψ is restricted to the lowest Landau level (5.4), the energy (5.2) is equal to

$$E(\psi) = \Omega + E_{\mathrm{LLL}}(\psi),$$

where E_{LLL} is given by (5.1). The aim of this chapter is to minimize $E_{\mathrm{LLL}}(\psi)$ in the space (5.4). The modulus of the wave function and the location of the zeros are plotted in Figure 1.7 for $\Omega = 0.999$. We will see that the characteristic size of the condensate, that is, the region where the wave function is significant (left part of Figure 1.7), is proportional to $(1-\Omega)^{-1/4}$. In this region, vortices are arranged on a triangular lattice, while outside, the wave function is of small amplitude, yet the analysis of the zeros is still of interest.

5.1.2 Construction of an upper bound

The mathematical description of the vortex structure involves the definition of a hexagonal or triangular lattice: in what follows, $\ell = \nu(\mathbf{Z} + \tau\mathbf{Z})$, with $\nu \in \mathbf{R}^+$ denoting a hexagonal lattice if $\tau = e^{2i\pi/3}$. The unit cell centered at the points of the lattice is called Q and has volume V. The symbol $⨍$ denotes the average of an ℓ-periodic function: $⨍ f = \frac{1}{V}\int_Q f$. Our main result is the construction of a test function that provides an upper bound for the energy:

Theorem 5.1. *There exists a sequence of functions ψ_Ω of the form (5.4) such that as Ω tends to* 1*,*

$$E_{\mathrm{LLL}}(\psi_\Omega) \sim \frac{2\sqrt{2}}{3}\sqrt{\frac{Nab}{\pi}(1-\Omega)}, \quad \textit{where } b = \frac{⨍ |\eta|^4}{(⨍ |\eta|^2)^2} \tag{5.5}$$

and $|\eta|$ is a periodic function that vanishes at each point of the lattice ℓ and $\eta(z)e^{\Omega|z|^2/2}$ is a holomorphic function.

The parameter b carries the averaged energy contribution of the vortex lattice in each cell, while the numerical coefficient in front of the square root is due to the shape of the slowly varying profile of the wave function. The parameter b is called the Abrikosov parameter [2, 93], since it arises in the study of superconductors near the second critical field. An approximate value is 1.16. In fact, b minimizes the ratio given in (5.5) among all possible lattices and functions having the properties of η [93, 8]. The function η is explicit, as we will see below.

The construction of the test function is inspired by the numerical computations in [7]: we minimize the energy (5.1) for test functions (5.4). We use a description of the polynomials P by its roots z_i and find the optimal location of the z_i through a conjugate gradient method. It provides the following pattern for vortices (see Figure 1.7): in a central region, vortices are located on a regular triangular lattice, while the lattice is distorted towards the edges. The density plot of $|\psi|$ shows that the only visible vortices are the ones in the regular lattice part, the outer ones being in a region of very low density. Our test function in Theorem 5.1 is intended to reproduce this pattern.

Regular lattice

Let us first explain, for the case of a regular lattice, how a large number of zeros in the test function can modify its decay at infinity: if the vortices are located on a regular lattice, the wave function decays like a Gaussian and we provide a rigorous proof of the energy estimate obtained by Ho [79].

Theorem 5.2. *Let ℓ be a regular hexagonal lattice, Q a unit cell, and $V = |Q| > \pi/\Omega$. Let*

$$\psi_R(z) = A_R \prod_{j\in\ell\cap B_R} (z-j)e^{-\Omega|z|^2/2} \tag{5.6}$$

with A_R chosen such that $\|\psi_R\|_{L^2(\mathbf{R}^2)} = 1$. Then as R tends to ∞,

$$|\psi_R(z)| \to \psi(z) = \frac{1}{\sqrt{\pi}\sigma}|\eta(z)|e^{-|z|^2/(2\sigma^2)} \text{ in } L^p\left(\mathbf{R}^2, (1+|z|^2)dz\right) \tag{5.7}$$

for all $p \geq 1$, where

$$\frac{1}{\sigma^2} = \Omega - \frac{\pi}{V} \tag{5.8}$$

and η is as defined in Theorem 5.1. Moreover, $|\eta|$ satisfies $-\Delta(\log|\eta|) = 2\pi\delta_0 - 2\pi/V$ in Q, with periodic boundary conditions. In addition, $\lim_{R\to\infty} E_{\mathrm{LLL}}(\psi_R) = E_{\mathrm{LLL}}(\psi)$. As σ tends to infinity, then

$$E_{\mathrm{LLL}}(\psi) \sim \frac{(1-\Omega^2)}{2}\sigma^2 + \frac{1}{4}\frac{Nab}{\pi\sigma^2}. \tag{5.9}$$

The main feature of the periodic lattice is to modify the decay of the Gaussian from $e^{-\Omega|z|^2/2}$ to $e^{-|z|^2/2\sigma^2}$, where σ depends on the volume through (5.8). We need to choose the optimal σ in (5.9), which yields

$$\sigma^4(1-\Omega^2) = \frac{1}{2}\frac{Nab}{\pi}. \tag{5.10}$$

This value of σ indeed satisfies $\sigma \to \infty$ as Ω tends to 1. The volume condition (5.8) matched with the value of σ in (5.10) implies $V = \pi\left(\Omega + \sqrt{2\pi(1-\Omega^2)/(Nab)}\right)$. The estimate of the energy is thus

$$E_{\mathrm{LLL}}(\psi) \underset{\Omega\to 1}{\sim} \sqrt{\frac{Nab}{\pi}(1-\Omega)}. \tag{5.11}$$

This is to be compared to (5.5), which is better by a factor $\sqrt{8/9}$, but is of the same magnitude, since $1-\Omega$ is small. Let us emphasize the presence of the coefficient b: it takes into account the averaged vortex contribution on each cell. As in the case of superconductors near H_{c_2}, for the Abrikosov lattice, the optimal lattice minimizing the ratio b is the hexagonal one [93]. Note that our proof could hold with other lattices than the hexagonal one, but the corresponding coefficient b would be higher.

The function η is explicit and related to the Abrikosov problem [2, 93]:

$$\eta(z) = e^{\Omega z^2/2} e^{-\Omega|z|^2/2}\Theta(\lambda z, e^{2i\pi/3}), \quad \text{where } \lambda = \sqrt{\frac{\Omega\sqrt{3}}{2\pi}}, \tag{5.12}$$

$$\Theta(v,\tau) = \frac{1}{i}\sum_{n=-\infty}^{+\infty}(-1)^n e^{i\pi\tau(n+1/2)^2} e^{(2n+1)\pi i v}, \quad v \in \mathbf{C}\,. \tag{5.13}$$

Given the invariance properties of the Theta function on a lattice, $|\eta|$ is periodic on the lattice $\lambda\mathbf{Z} + \lambda e^{2i\pi/3}\mathbf{Z}$. It can be proved that all functions such that $\eta e^{\Omega|z|^2}$ is holomorphic and whose modulus is periodic on a lattice are given by (5.12). A proof that the triangular lattice, that is, $\tau = e^{2i\pi/3}$ minimizes the ratio $b = ⨍ |\eta|^4/(⨍ |\eta|^2)^2$ for such functions η can be found in [8].

Distorted lattice and inverted parabola

The main observation is that modifying the location of the vortices from a regular lattice can change the decay of the wave function and hence improve the energy estimate. This decay is similar to that of an inverted parabola, as already observed by [49, 141, 154], and we are going to justify this observation.

If one considers the minimization of the energy (5.1) without the constraint (5.4), one finds that $Na|\psi|^2$ is equal to the positive part of $\mu - (1-\Omega^2)|z|^2/2$, where μ is the Lagrange multiplier due to the constraint $\int |\psi|^2 = 1$; $|\psi|$ is thus the inverted parabola

$$|\psi_{\min}(r)|^2 = \frac{2}{\pi R_0^2}\left(1 - \frac{r^2}{R_0^2}\right)\mathbf{1}_{r<R_0}, \quad R_0 = \left(\frac{4Na}{\pi(1-\Omega^2)}\right)^{1/4}. \tag{5.14}$$

The reduced energy is

$$\epsilon_{\min} = E_{\mathrm{LLL}}(\psi_{\min}) \sim \frac{2\sqrt{2}}{3}\sqrt{\frac{Na(1-\Omega)}{\pi}}\,. \tag{5.15}$$

This is close to the upper bound (5.5); the numerical factor is the same except for the coefficient b, coming from the averaged vortex contribution. But if ψ is of the form (5.4), it cannot approximate (5.14) since $\psi_{\min} e^{\Omega|z|^2/2}$ is not a holomorphic function. Nevertheless, as Ω tends to 1, with an appropriate location of the zeros z_i of the polynomial P, the aim is to build a test function that provides a weak-star approximation of $\psi_{\min}$, and whose energy can be of the same order as (5.15) but with a coefficient $\sqrt{b}$ coming from the contribution of the lattice.

Ideas of the proofs

Let us now explain the main ideas of the proof developed in [4]. For the regular lattice, we split $\log|\psi_R(z)|$ into $v_R(z) + w_R(z)$ with

$$v_R(z) = \sum_{j \in l \cap B_R} \log|z-j| - \frac{1}{V}\int_Q \log|z-y-j|\,dy, \tag{5.16}$$

$$w_R(z) = \log A_R - \Omega\frac{|z|^2}{2} + \frac{1}{V}\sum_{j \in l \cap B_R}\int_Q \log|z-y-j|\,dy. \tag{5.17}$$

At this stage, we have just added and subtracted the sum of the integrals. As R tends to ∞, we prove that v_R converges to a periodic series v and e^{w_R} to a Gaussian with modified decay $1/\sigma^2$. The computation of the energy uses the double-scale convergence [19], which allows us to separate the integrals in v and the Gaussian and get the contribution of b.

Let us be more precise about Theorem 5.1. We perform a general transformation of the lattice in the following way: for j in ℓ, a regular triangular lattice of unit cell with volume $V = \pi/\Omega$, we define the transformed lattice ℓ'_R by

$$k \in \ell'_R \text{ if } k = \nu_R(|j|)\, j \text{ for } j \in \ell \cap B_R. \tag{5.18}$$

We assume that ν_R is close to 1 as Ω tends to 1, in the sense that

$$\nu_R^2(r) = 1 + \frac{f(r^2/R^2)}{R^2} + O\left(\frac{1}{R^4}\right) \text{ with } R = \left(\frac{4Nab}{\pi(1-\Omega^2)}\right)^{1/4}, \tag{5.19}$$

where $f(x)$ is a continuous function, such that for some γ, $f(\gamma) = \infty$ and $\int_0^\gamma f(s)\,ds = \infty$. Note that we do not take as inverted parabola (5.14), but we need to include the coefficient b in the radius.

We would like to apply the same proof as for the regular lattice, using v_R and w_R for this distorted lattice. In contrast to the proof for the regular lattice, we cannot study the two limits $R \to \infty$ in (5.7) and $\sigma \to \infty$ in (5.9) separately, since now R is related to Ω through (5.19). Hence, the lattice has a finite extent at each R and we have to pass to the limit in the double scale convergence at the same time as the scale of the lattice. We are unable to match w_R inside and outside the lattice and do the dominated convergence separately.

In order to circumvent this problem, we introduce an outer regular lattice, whose characteristic size tends to infinity in a last step. Let $\alpha \in (0, \gamma)$, R be related to Ω by (5.19), and

$$\lambda_R(r) = \begin{cases} \nu_R(r) & \text{if } r \leq \alpha R, \\ \nu_{\alpha,R} = \nu_R(\alpha R) & \text{if } r > \alpha R, \end{cases} \tag{5.20}$$

and $\ell'_R = \{\lambda_R(|j|)j : \quad j \in \ell\}$. For fixed α, we let R tend to ∞, and study the limit of the wave functions vanishing at each point of ℓ'_R:

$$\psi_R(z) = A_R \prod_{j \in \ell} (z - \lambda_R(|j|)j)\, e^{-\Omega\frac{|z|^2}{2}}. \tag{5.21}$$

Since α is fixed, $\nu_R(\alpha R)$ tends to 1. We use similar ideas as in the regular lattice case and identify a double-scale convergence to a periodic part on the one hand and a profile depending on the transformation f on the other hand, given by

$$|\psi(z)|^2 = |\eta(z)|^2 \left(e^{-F(|z|^2)} \mathbf{1}_{B_\alpha}(|z|) + e^{\alpha^2 f(\alpha^2) - F(\alpha^2) - f(\alpha^2)|z|^2} \mathbf{1}_{B_\alpha^c}(|z|) \right), \tag{5.22}$$

where F is a primitive of f. The proof uses as a main tool that λ_R is close to the identity. As a final step only, once we have passed to the limit $\Omega \to 1$, we let α tend to γ, so that the exterior regular lattice has a unit-cell volume that tends to infinity and the outer contribution disappears. We find an estimate for the energy:

$$E_{\mathrm{LLL}}(\psi_\Omega) \sim_{\Omega \to 1} \sqrt{\frac{2Nab(1-\Omega)}{\pi}} \int_0^\gamma \left(s e^{-F(s)} + \frac{1}{4} e^{-2F(s)} \right) ds, \tag{5.23}$$

where F is a primitive of f such that $\int_0^\gamma e^{-F(s)}\, ds = 1$.

We want to find which type of distortion f provides the optimal energy. The minimizer of (5.23) under $\int_0^\gamma e^{-F(s)}\, ds = 1$ is reached when

$$\gamma = 1 \text{ and } e^{-F(r^2)} = 2(1 - r^2). \tag{5.24}$$

Thus, the decay of the wave function is asymptotically an inverted parabola. The corresponding value of f is $f(s) = 1/(1-s)$. The limiting value of the energy is (5.5).

Let us point out that the proof uses two lattices: an initial regular lattice and an image lattice obtained by (5.18) and (5.19). The meaning of $\gamma = 1$ in (5.24) is that the initial lattice is truncated in the ball B_R: the points outside B_R are sent to infinity. There are two regions in the initial lattice: the points sufficiently far away from the circle of radius R, for which ν_R is almost one, and the points close to the circle, at distance less than $\sqrt{R}$, for instance. For the first category of points, the image lattice is an almost regular lattice and the image points are inside the disk B_R. These are the visible points on the density profile. In contrast, the points close to the circle are strongly modified by (5.19) and sent far away. This allows us to understand better the distorted shape of Figure 1.7. It turns out that R is both the critical radius for the initial lattice and the radius of the limiting inverted parabola (5.24).

For each R, this analysis gives an estimate of the number of necessary points in the distorted lattice: it is the number of cells in a regular lattice of unit volume

π included in a ball of radius R. Nevertheless, adding points in a far-away region modifies the energy by lower-order terms. We will see in the next section that the minimizer does not have a finite number of zeros but an infinite one.

One of our technical tools in the proof is to use an outer regular lattice whose spacing tends to infinity in a last step. If one wanted to get rid of this trick, one would need to count the number of points in the lattice closest to the limiting circle of radius R and estimate the convergence of v_R and w_R due to the fact that these limiting points do not lie on a circle but on the edges of hexagons. We are not able to prove that the finite extension of the lattice (which becomes infinite with an outer regular lattice) does not create a boundary contribution in the energy. These boundary effects seem to be more important than we expected, and are related to known problems about counting the number of points of a lattice in an annulus. Moreover, given the fact that the minimizer has an infinite number of zeros, points are needed in the outer region.

5.1.3 Properties of the minimizer

In order to derive properties of the minimizers of (5.1) in the space (5.4), we need to write an equation satisfied by the minimizers. This requires the knowledge of the projector onto our space of minimization. There is a natural Hilbert space related to our minimization problem, the so called Fock–Bargmann space [8, 26, 65, 110]:

$$\mathcal{F} = \left\{ f \text{ is holomorphic, } \int_{\mathbf{R}^2} |f|^2 e^{-\Omega |z|^2} < \infty \right\}. \tag{5.25}$$

This space is a Hilbert space endowed with the scalar product $\langle f, g \rangle_{\mathcal{F}} = \int_{\mathbf{R}^2} f(z) g(z) e^{-\Omega|z|^2}$. The projection of a general function $g(z, \bar{z})$ onto $\mathcal{F}$ is explicit:

$$\Pi(g) = \frac{\Omega}{\pi} \int_{\mathbf{R}^2} e^{\Omega z \overline{z'}} e^{-\Omega |z'|^2} g(z', \bar{z}'). \tag{5.26}$$

If g is a holomorphic function, then an integration by parts yields $\Pi(g) = g$. Using this expression, we are able to derive an equation for the minimizer:

Theorem 5.3. *There exists a minimizer of (5.1) with the constraint that $f(z) = \psi(z) e^{\Omega|z|^2/2}$ belongs to $\mathcal{F}$. Moreover, f is a solution of the following equation:*

$$\Pi\Big(\Big(\frac{(1-\Omega^2)}{2} |z|^2 + Na|f|^2 e^{-\Omega|z|^2} - \mu \Big) f \Big) = 0, \tag{5.27}$$

where μ is the Lagrange multiplier coming from the L^2 constraint.

The proof of this theorem will not be included here. It relies on a precise knowledge of the compact embedding of some more regular spaces into $\mathcal{F}$ [44]. We refer to [9] for the proof. The equation (5.27) allows us to derive that the minimizer cannot be a polynomial:

Theorem 5.4. *If $f \in \mathcal{F}$ is such that $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ minimizes (5.1), and $(1-\Omega)$ is small, then f has an infinite number of zeros.*

The proof relies on an explicit formulation of (5.27) and a contradiction argument on the number of zeros if it is finite.

The tools introduced here can provide another proof of Theorem 5.1. Namely, if η is the periodic function on the lattice introduced in (5.12) and p is the inverted parabola

$$p(z) = \sqrt{\frac{2}{\pi R^2}\left(1 - \frac{|z|^2}{R^2}\right) 1_{\{|z| \le R\}}}, \quad R = \left(\frac{2Nab}{\pi(1-\Omega)}\right)^{1/4}, \tag{5.28}$$

then $\Pi(p(z)\eta(z))e^{-\Omega|z|^2/2}$ is an appropriate test function that reproduces the same upper bound as (5.5) as proved in [9]. This test function is in fact close to $p(z)\eta(z)e^{-\Omega|z|^2/2}$. A number of open questions arise, in particular we would like to prove that $\Pi(p(z)\eta(z))e^{-\Omega|z|^2/2}$ is a good approximation of the minimizer. This will be described in the open problem section.

5.1.4 Other trapping potentials

Let us point out that other trapping potentials than r^2 can be dealt with using these techniques. In [7], we have addressed the case of $r^2 + kr^4$ with k small, following recent experiments [40, 150]. According to the values of Ω, a giant vortex can be obtained. This will be detailed in the last section.

The chapter is organized as follows: we prove Theorems 5.2 and 5.1 in the first two sections, then Theorem 5.4 in Section 5.4. Finally, we address the issue of other trapping potentials and describe some open questions.

5.2 Regular lattice

In this section, we prove Theorem 5.2. We first need two technical lemmas:

Lemma 5.5. *Let ℓ be a lattice, and denote by Q its unit cell centered at 0. Let $Q_R = \bigcup_{k \in \ell \cap B_R}(Q + k)$ and for x in $\mathbf{R}^2$, let*

$$h_R(x) = \int_{Q_R} \left(\log|x - x'| - \log|x'|\right) dx'.$$

Then there exist $C > 0$ and $R_0 > 0$ such that

$$\forall R \ge R_0, \quad h_R(x) \le \left(\frac{\pi}{2} + \frac{C}{R}\right)|x|^2.$$

Proof: If Q_R was a ball, then the integral could be computed explicitly. Thus, we use a ball close to Q_R and estimate the difference. We separate the integral defining h_R into two parts:

$$h_R(x) = \int_{B_{R-a}} \big(\log|x - x'| - \log|x'|\big)\, dx' + \int_{Q_R\setminus B_{R-a}} \big(\log|x - x'| - \log|x'|\big)\, dx',$$

where $a > 0$ is independent of R and such that $B_{R-a} \subset Q_R$. The first term is the radial solution of $\Delta u = \mathbf{1}_{B_{R-a}}$ such that $u(0) = 0$. One easily computes this solution:

$$u(x) = \frac{\pi}{2}|x|^2\mathbf{1}_{B_{R-a}} + \pi(R-a)^2\left(\frac{1}{2} + \log\left(\frac{|x|}{R-a}\right)\right)\mathbf{1}_{B^c_{R-a}}.$$

Next, we consider the second term defining h_R and use the inequality $\log(t) \le \frac{1}{2}(t^2 - 1)$, valid for any $t > 0$:

$$\begin{aligned}\int_{Q_R\setminus B_{R-a}} \big(\log|x - x'| - \log|x'|\big)\, dx' &\le \int_{Q_R\setminus B_{R-a}} \frac{1}{2}\left(\frac{|x-x'|^2}{|x'|^2} - 1\right) dx' \\ &= |x|^2 \int_{Q_R\setminus B_{R-a}} \frac{dx'}{2|x'|^2} \le C\frac{|x|^2}{R},\end{aligned}$$

the constant C being independent of R and x. Collecting both results, we infer that

$$\begin{aligned}h_R(x) &\le \frac{\pi}{2}|x|^2\mathbf{1}_{B_{R-a}}(x) + \pi(R-a)^2\left(\frac{1}{2} + \log\left(\frac{|x|}{R-a}\right)\right)\mathbf{1}_{B^c_{R-a}}(x) + \frac{C}{R}|x|^2 \\ &\le \left(\frac{\pi}{2} + \frac{C}{R}\right)|x|^2,\end{aligned}$$

using here again $\log(t) \le \frac{1}{2}(t^2 - 1)$. This gives the result. □

Lemma 5.6. *Let ℓ be the hexagonal lattice, and let Q be its elementary unit cell (i.e., the regular hexagon centered at* 0*). Let*

$$g(x) = \log|x| - \frac{1}{|Q|}\int_Q \log|x - y|\, dy. \tag{5.29}$$

Then we have, for some constant $C > 0$,

$$\forall x \in B_1^c, \quad |g(x)| \le \frac{C}{|x|^3}. \tag{5.30}$$

Hence, the function

$$v(x) = \sum_{j\in\ell} g(x - j) \tag{5.31}$$

is such that $e^{v(x)}$ exists, is continuous on $\mathbb{R}^2$, and is ℓ-periodic.

Proof: We first point out that g is continuous on $\mathbb{R}^2 \setminus \{0\}$. Hence, we need to show (5.30) only on B_a^c for some $a > 0$. We fix $a > 0$ such that $Q \subset B_{\frac{a}{2}}$. For any $x \in B_a^c$ and any $y \in Q$, we have $\frac{|x-y|}{|x|} \geq \frac{|x|-\frac{a}{2}}{|x|} \geq \frac{1}{2}$. Hence,

$$\frac{|x-y|^2}{|x|^2} - 1 \geq -\frac{3}{4}.$$

For any $t > -\frac{3}{4}$, we have

$$t - \frac{t^2}{2} - |t|^3 \leq \log(1+t) \leq t - \frac{t^2}{2} + \frac{t^3}{3}.$$

Hence, writing $g(x) = -\frac{1}{|Q|} \int_Q \frac{1}{2} \log\left(1 - \frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right) dy$, we infer that

$$\frac{1}{2|Q|}\int_Q\left(-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2} - \frac{1}{2}\left(-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^2 - \left|-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right|^3\right) dy \leq -g(x)$$

$$\leq \frac{1}{2|Q|}\int_Q\left(-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2} - \frac{1}{2}\left(-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^2 + \frac{1}{3}\left(-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^3\right) dy.$$

Since Q is symmetric with respect to the origin, $\int_Q y \cdot x \, dy = 0$. In addition, Q is invariant under the rotation of angle $\pi/3$, so one easily shows that $\int_Q \left(|y|^2 - 2(y\cdot x)^2\right) dy = 0$. We thus have

$$\frac{1}{2|Q|}\int_Q\left(\frac{2y\cdot x|y|^2}{|x|^4} - \frac{|y|^4}{|x|^4} - \left|-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right|^3\right) dy \leq g(x)$$

$$\leq \frac{1}{2|Q|}\int_Q\left(\frac{2y\cdot x|y|^2}{|x|^4} - \frac{|y|^4}{|x|^4} + \frac{1}{3}\left(-\frac{2y\cdot x}{|x|^2} + \frac{|y|^2}{|x|^2}\right)^3\right) dy.$$

Using $|y| \leq \frac{a}{2}$, we end up with

$$|g(x)| \leq \frac{1}{|Q|}\int_Q\left(\frac{|y|^3}{|x|^3} + \frac{|y|^4}{2|x|^4} + \frac{1}{2}\left|\frac{2|y|}{|x|} + \frac{|y|^2}{|x|^2}\right|^3\right) dy \leq \frac{\frac{a^3}{8} + \frac{a^3}{16} + 2a^3}{|x|^3}.$$

This ensures that the series (5.31) converges normally on any set of the form $\left(\bigcup_{j\in\ell} B_\varepsilon(j)\right)^c$, which implies that v exists, is ℓ-periodic, and is continuous on $\mathbb{R}^2 \setminus \ell$. Near a point $k \in \ell$, we write

$$e^{v(x)} = |x-k| e^{-\frac{1}{|Q|}\int_Q \log|x-y|\, dy} e^{\sum_{j\in\ell\setminus\{k\}} g(x-j)},$$

and the conclusion follows.

Remark 5.7. It is in this lemma that we have used the symmetry properties of the lattice. However, the same proof applies to a general lattice (which does not necessarily have the above symmetries). In this case, one needs to use instead of $g = \log|\cdot| * (\delta_0 - \frac{1}{|Q|}\mathbf{1}_Q)$ the convolution of $\log|\cdot|$ with a distribution g_0 such that $\sum_{k\in\ell} g_0(x-k) = \sum_{k\in\ell}\delta_k - \frac{1}{|Q|}$, and such that the first harmonic moments of g_0 cancel. This is always possible (in such a case g_0 is not supported inside Q).

Remark 5.8. The estimate (5.30) is valid for a fixed hexagonal lattice ℓ_0. Now, $g = g_\ell$ depends on ℓ in the following way: if $\ell = \lambda\ell_0$, then $g_\ell(z) = g_{\ell_0}\left(\frac{z}{\lambda}\right)$. Hence, $|g_\ell(z)| \le \frac{C_0\lambda^3}{|z|^3}$ if $|z| \ge \lambda$, for some constant C_0 independent of ℓ.

Proof of Theorem 5.2: Let $f_R(z) = \log|\psi_R(z)|$. We split f_R into

$$f_R(z) = v_R(z) + w_R(z) \tag{5.32}$$

with

$$v_R(z) = \sum_{j\in l\cap B_R} \log|z-j| - \frac{1}{V}\int_Q \log|z-y-j|\,dy, \tag{5.33}$$

$$w_R(z) = \log A_R - \Omega\frac{|z|^2}{2} + \frac{1}{V}\sum_{j\in l\cap B_R}\int_Q \log|z-y-j|\,dy. \tag{5.34}$$

Let v be given by (5.31). We have

$$v_R(z) - v(z) = \sum_{j\in\ell\cap B_R^c} g(z-j).$$

Hence, if $z\in B_R$, we deduce from Lemma 5.6 that

$$|v_R(z) - v(z)| \le \sum_{j\in\ell\cap B_R^c} \frac{C}{|z-j|^3}$$

for some constant C independent of R and z. One can thus find a constant C independent of R such that

$$\forall A\in(0,R), \quad \|v_R - v\|_{L^\infty(B_{R-A})} \le \frac{C}{A}. \tag{5.35}$$

In addition, we have, for any $z\in\mathbf{R}^2$, denoting by j_z the unique point of ℓ such that $|j_z - z| < 1$,

$$v_R(z) \le \log|z-j_z| + C + \sum_{j\in\ell\setminus\{j_z\}} \frac{C}{|z-j|^3} \le \log|z-j_z| + C,$$

for various constants C independent of z and R. Hence, e^{v_R} is bounded in $L^\infty(\mathbf{R}^2)$ independently of R. Next, using the inequality $|e^a - e^b| \le \frac{1}{2}(e^a+e^b)|a-b|$ and (5.35), we infer that e^{v_R} converges to e^v in $L^\infty_{\mathrm{loc}}(\mathbf{R}^2)$.

Let us set $\tilde{w}_R(z) = w_R(z) - w_R(0) - \log(A_R) + \frac{1}{2\sigma^2}|z|^2$. Applying Lemma 5.5, we have

$$\tilde{w}_R(z) \le -\Omega\frac{|z|^2}{2} + \left(\frac{\pi}{2V} + \frac{C}{R}\right)|z|^2 + \frac{1}{2\sigma^2}|z|^2 = \frac{C}{R}|z|^2. \tag{5.36}$$

In addition, $\tilde{w}_R$ is a harmonic function in $Q_R = \bigcup_{j\in\ell\cap B_R}(Q+j)$ and vanishes at 0. Hence, using the Harnack inequality, $\tilde{w}_R$ is bounded and we may extract convergence of $\tilde{w}_R$ in $L^\infty_{\mathrm{loc}}(\mathbf{R}^2)$ to some $\tilde{w}$, which is harmonic, nonpositive, and vanishes at 0. Applying Liouville's theorem, we find that $\tilde{w} = 0$. Gathering all the previous results, we thus have

$$\frac{|\psi_R(z)|}{A_R e^{w_R(0)}} \longrightarrow e^{v(z)}e^{-|z|^2/(2\sigma^2)} \quad \text{almost everywhere in } \mathbb{R}^2. \tag{5.37}$$

For R large enough, (5.36) also implies

$$\frac{|\psi_R(z)|}{A_R e^{w_R(0)}} \le Ce^{-\frac{|z|^2}{4\sigma^2}}. \tag{5.38}$$

From (5.37) and (5.38), we apply the dominated convergence theorem and find that as R tends to infinity,

$$\frac{|\psi_R|}{A_R e^{w_R(0)}} \longrightarrow e^{v(z)}e^{-|z|^2/(2\sigma^2)} \text{ in } L^p(\mathbb{R}^2), \quad \forall p \ge 1.$$

Using the fact that $\|\psi_R\|_{L^2} = 1$, we deduce that $1/(A_R e^{w_R(0)})$ converges to the appropriate constant, so that (5.7) holds.

Then we write the limiting energy, z being identified with a vector in $\mathbf{R}^2$:

$$\begin{aligned} E_{\mathrm{LLL}}(\psi) &= \int_{\mathbf{R}^2}\left(\frac{(1-\Omega^2)}{2}|z|^2|\eta(z)|^2 e^{-\frac{|z|^2}{\sigma^2}} + \frac{Na}{2\sigma^2}|\eta(z)|^4 e^{-\frac{2|z|^2}{\sigma^2}}\right)\frac{dz}{\pi\sigma^2} \\ &= \int_{\mathbf{R}^2}\left(\frac{(1-\Omega^2)}{2}\sigma^2|\eta(\sigma\xi)|^2 e^{-|\xi|^2} + \frac{Na}{2\sigma^2}|\eta(\sigma\xi)|^4 e^{-2|\xi|^2}\right)d\xi. \end{aligned}$$

The function η is periodic, so $|\eta(\sigma\xi)|^2$ and $|\eta(\sigma\xi)|^4$ respectively converge L^∞-weak-* to $⨍|\eta|^2$ and $⨍|\eta|^4$ (see [19]). Hence, we obtain (5.9).

5.3 Distorted lattice

In this section, we prove two theorems, that will imply Theorem 5.1. The first theorem consists in studying a distorted lattice and finding the limit of the wave function with an infinite number of vortices. The proof is similar to the regular lattice case, since only the central vortices are displaced from their regular location. The second theorem consists in letting Ω tend to 1 and using the double-scale convergence. The proof is more involved since this is precisely where the distortion of the lattice appears.

Theorem 5.9. *Let ℓ be a hexagonal lattice, and let Q be the regular hexagon of area π centered at zero. Let $\gamma > 0$ and let f be a positive Lipschitz continuous function defined in $[0, \gamma)$ such that*

$$\lim_{t \to \gamma} f(t) = +\infty \quad \text{and} \quad \lim_{t \to \gamma} \int_0^t f(s)ds = \infty. \tag{5.39}$$

Let us define

$$\nu_R(t) = 1 + \frac{1}{2R^2} f\left(\frac{t^2}{R^2}\right) + O\left(\frac{1}{R^4}\right), \tag{5.40}$$

where $O\left(\frac{1}{R^4}\right)$ is uniform with respect to $t \in \mathbf{R}^+$. Let $\alpha \in (0, \gamma)$ and define

$$\lambda_R(t) = \begin{cases} \nu_R(t) & \text{if } t \le \alpha R, \\ \nu_{\alpha,R} = \nu_R(\alpha R) & \text{if } t > \alpha R. \end{cases} \tag{5.41}$$

For $R' > R$, we define

$$\psi_{R,R'}(z) = A_{R,R'} \prod_{j \in \ell \cap B_{R'}} (z - \lambda_R(|j|)j)\, e^{-\Omega \frac{|z|^2}{2}}, \tag{5.42}$$

where $A_{R,R'}$ is such that $\|\psi_{R,R'}\|_{L^2(\mathbf{R}^2)} = 1$. Then, we have the following convergence in $L^p(\mathbf{R}^2, (1+|x|^2)dx)$, for any $p < +\infty$:

$$\left|\psi_{R,R'}\right| \underset{R' \to +\infty}{\longrightarrow} A_R e^{v_R\left(\frac{z}{\nu_{\alpha,R}}\right) + w_R(z) + \left(\frac{1}{\nu_{\alpha,R}{}^2} - \Omega\right)\frac{|z|^2}{2}}, \tag{5.43}$$

where

$$w_R(z) = \sum_{j \in \ell \cap B_{\alpha R}} \frac{1}{\nu_{\alpha,R}^2 |Q|} \int_{\nu_{\alpha,R} Q} \log\left(\frac{|z - y - \nu_R(|j|)j|}{|z - y - \nu_{\alpha,R} j|}\right) dy, \tag{5.44}$$

and

$$v_R(z) = \sum_{j \in \ell} g\left(z - \frac{\lambda_R(|j|)}{\nu_{\alpha,R}} j\right), \tag{5.45}$$

the function g being defined by (5.29).

Then, we let Ω tend to 1, or equivalently R to infinity:

Theorem 5.10. *With the same definitions as in Theorem 5.9, we have*

$$\forall n \ge 1, \quad e^{n v_R(Rz)} \underset{R \to +\infty}{\overset{*}{\rightharpoonup}} \fint e^{nv} \quad \text{in} \quad L^\infty(\mathbf{R}^2), \tag{5.46}$$

$$e^{2w_R(Rz)+\left(\frac{1}{\lambda_R(\alpha R)^2}-1\right)R^2|z|^2} \underset{R\to+\infty}{\longrightarrow} \rho(z) \tag{5.47}$$

in $L^p(\mathbf{R}^2, (1+|x|^2)dx)$, $\forall p \geq 1$, *where, v is given by (5.31),*

$$\rho(z) = e^{-F(|z|^2)}\mathbf{1}_{B_\alpha}(|z|) + e^{\alpha^2 f(\alpha^2)-F(\alpha^2)-f(\alpha^2)|z|^2}\mathbf{1}_{B_\alpha^c}(|z|), \tag{5.48}$$

and F is a primitive of f such that $⨍ e^{2v} \int \rho = 1$.

Proof of Theorem 5.1: We let Ω tend to 1 and R be given by (5.19), and we take a diagonal sequence in R'. Theorems 5.9, 5.10 and double-scale convergence [19] provide the convergence of $\int |\psi_{R,R'}(Rz)|^2$ to $⨍ e^{2v} \int \rho$, and similarly for the energy, $E_{\mathrm{LLL}}(\psi_{R,R'}(Rz))$ is equivalent to

$$\sqrt{\frac{2Nab(1-\Omega)}{\pi}}\left(⨍ e^{2v}\int_0^\infty s\rho(\sqrt{s})\,ds + \frac{1}{4}⨍ e^{4v}\int_0^\infty \rho^2(\sqrt{s})\,ds\right), \tag{5.49}$$

where F is a primitive of f such that $\int_0^\gamma e^{-F(s)}\,ds = 1$ and $⨍ e^{2v} = 1$. If one lets α tend to γ, the contribution to ρ in the outer part B_γ^c vanishes and the energy is given by (5.23).

We want to find which type of distortion f provides the optimal energy. The minimizer of (5.23) under $\int_0^\gamma e^{-F(s)}\,ds = 1$ is reached when

$$\gamma = 1 \text{ and } e^{-F(r^2)} = 2(1-r^2). \tag{5.50}$$

Thus, the decay of the wave function is asymptotically an inverted parabola. The corresponding value of f is $f(s) = 1/(1-s)$. The limiting value of the energy is (5.5).

Proof of Theorem 5.9: This proof is a mere adaptation of Section 5.2. Indeed, up to normalization by a constant, the function $\log|\psi_{R,R'}|^2$ is equal to

$$\log\left|\psi_{R,R'}(x)\right|^2 = 2\sum_{j\in\ell\cap B_{R'}}\Bigg(\log|x-\lambda_R(|j|)j| \tag{5.51}$$

$$-\frac{1}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\log|x-y-\lambda_R(|j|)j|dy\Bigg) \tag{5.52}$$

$$+\sum_{j\in\ell\cap B_{R'}}\frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q}\log|x-y-\lambda_R(|j|)j|dy \tag{5.53}$$

$$-|x|^2. \tag{5.54}$$

The sum (5.51)–(5.52) may be written

$$\sum_{j\in\ell\cap B_{R'}} g\left(\frac{x}{\nu_{\alpha,R}} - \frac{\lambda_R(|j|)}{\nu_{\alpha,R}}j\right), \tag{5.55}$$

where g is defined by (5.29). Now, R being fixed, Lemma 5.6 ensures that the above sum converges as R' goes to infinity to $v_R\left(\frac{x}{\nu_{\alpha,R}}\right)$, where v_R is defined by (5.45). Moreover, the convergence of the exponential of (5.55) to $e^{v_R\left(\frac{x}{\nu_{\alpha,R}}\right)}$ is the same as in Theorem 5.2, that is, $L^\infty_{\text{loc}}(\mathbf{R}^2)$. Next, the sum (5.53) is equal to

$$\begin{aligned}
&\sum_{j\in\ell\cap B_{R'}} \frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q} \log|x-y-\lambda_R(|j|)j|dy \\
&\quad= \sum_{j\in\ell\cap B_{\alpha R}} \frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q} \log\frac{|x-y-\nu_R(|j|)j|}{|x-y-\nu_{\alpha,R}\ j|}dy \\
&\quad+ \sum_{j\in\ell\cap B_{R'}} \frac{2}{\nu_{\alpha,R}{}^2|Q|}\int_{\nu_{\alpha,R}Q} \log|x-y-\nu_{\alpha,R}\ j|dy.
\end{aligned} \tag{5.56}$$

The first sum in the left-hand side of (5.56) is $w_R(x)$, while the second sum is the one appearing in (5.34), with $\nu_{\alpha,R}\ell$ replacing ℓ. Since this lattice is also a hexagonal one (with a different volume for its unit cell), the proof of its convergence applies, using Lemma 5.5.

Proof of Theorem 5.10: For simplicity, we will give the proof in the case that the $O(1/R^4)$ is zero. We start with the proof of (5.47). We define $\varepsilon > 0$ depending on R such that, as R tends to infinity,

$$\begin{cases} R\varepsilon \longrightarrow +\infty, \\ R\varepsilon^2 \longrightarrow 0. \end{cases} \tag{5.57}$$

For instance, $\varepsilon = R^{-\frac{3}{4}}$ is a suitable choice. We write (recall that here $z = x + iy$ and dz denotes the Lebesgue measure $dx\,dy$)

$$w_R(Rx) = \sum_{k\in\frac{\ell}{R}\cap B_\alpha} \frac{R^2}{\nu_{\alpha,R}{}^2|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q} \log\left(\frac{|x-z-\nu_R(R|k|)k|}{|x-z-\nu_{\alpha,R}k|}\right)dz,$$

we split this sum into terms for which $|k-x| < \varepsilon$ and terms for which $|x-k| \geq \varepsilon$: in the first case, we use the inequality

$$\forall a, b > 0, \quad |\log a - \log b| \leq \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)|b-a|$$

and the fact that $|\nu_{\alpha,R} - \nu_R(|k|R)| \leq \frac{C}{R^2}$ for some constant C independent of R and x. Hence,

$$\begin{aligned}
&\left|\sum_{|k-x|<\varepsilon} \frac{R^2}{\nu_{\alpha,R}|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q} \log\left(\frac{|x-z-\nu_R(R|k|)k|}{|x-z-\nu_{\alpha,R}k|}\right)dz\right| \\
&\quad\leq \sum_{|k-x|<\varepsilon} \frac{R^2}{2\nu_{\alpha,R}|Q|}\int_{\frac{\nu_{\alpha,R}}{R}Q} \left(\frac{1}{|x-z-\nu_R(R|k|)k|}\right.
\end{aligned}$$

$$+ \frac{1}{|x - z - \nu_{\alpha,R} k|}\Big)|k||\nu_{\alpha,R} - \nu_R(|k|R)|dz$$

$$\leq \frac{C}{R^2} \sum_{|k-x|<\varepsilon} \frac{R^2}{2\nu_{\alpha,R}|Q|} \int_{B_{\frac{3}{R}}} \frac{dy}{|y|} \leq C\# \left(\frac{\ell}{R} \cap B_\varepsilon(x)\right) \frac{1}{R} = CR\varepsilon^2,$$

which tends to zero as $R \to +\infty$. Next, we deal with $|k - x| \geq \varepsilon$, and denote the corresponding sum by $T_R(x)$:

$$T_R(x) = \sum_{|k-x|\geq\varepsilon} \frac{R^2}{\nu_{\alpha,R}{}^2|Q|} \int_{\frac{\nu_{\alpha,R}}{R}Q} \log\left(\frac{|x - z - \nu_R(R|k|)k|}{|x - z - \nu_{\alpha,R}k|}\right) dz.$$

Using the equality $\nu_R(R|k|) = 1 + \frac{f(|k|^2)}{2R^2}$, valid for any $|k| \leq \alpha$, we deduce that

$$T_R(x) = \sum_{|k-x|\geq\varepsilon} \frac{R^2}{\nu_{\alpha,R}{}^2|Q|} \int_{\frac{\nu_{\alpha,R}}{R}Q} \log\left(\frac{|x - z - k - \frac{f(|k|^2)}{2R^2}k|}{|x - z - k - \frac{f(\alpha^2)}{2R^2}k|}\right) dz.$$

We have $|x - z - k| \geq |x - k| - |z| \geq \varepsilon - \frac{C}{R} = \varepsilon(1 - \frac{C}{\varepsilon R})$ for $z \in \frac{\nu_{\alpha,R}}{R}Q$, so that for R large enough, we get $|x - z - k| \geq \frac{\varepsilon}{2}$. Hence, developing the quotient in the logarithm, we get

$$T_R(x) = \sum_{|k-x|\geq\varepsilon} \frac{R^2}{\nu_{\alpha,R}{}^2|Q|} \int_{\frac{\nu_{\alpha,R}}{R}Q} \log\left(\frac{1 - \frac{f(|k|^2)}{R^2|x-z-k|^2}k \cdot (x - z - k) + O\left(\frac{1}{\varepsilon^2 R^4}\right)}{1 - \frac{f(\alpha^2)}{R^2|x-z-k|^2}k \cdot (x - z - k) + O\left(\frac{1}{\varepsilon^2 R^4}\right)}\right) dz,$$

where the $O\left(\frac{1}{\varepsilon^2 R^4}\right)$ are uniform with respect to x and k. Developing the logarithm, we thus obtain

$$T_R(x) = \sum_{|k-x|\geq\varepsilon} \left(\frac{R^2}{\nu_{\alpha,R}{}^2|Q|} \int_{\frac{\nu_{\alpha,R}}{R}Q} \frac{f(\alpha^2) - f(|k|^2)}{R^2} \frac{k \cdot (x - z - k)}{|x - z - k|^2} dz\right) + O\left(\frac{1}{\varepsilon^2 R^2}\right).$$

Using the fact that f is smooth in $[0, \alpha]$, and recalling that the sum is a sum over the set $\frac{\ell}{R} \cap B_\alpha \cap B_\varepsilon(x)^c$, we find that it converges to the corresponding integral, namely

$$\lim_{R\to+\infty} T_R(x) = \frac{1}{|Q|} \int_{B_\alpha} \left(f(\alpha^2) - f(|y|^2)\right) \frac{y \cdot (x - y)}{|x - y|^2} dy.$$

We then point out that $\frac{x-y}{|x-y|^2} = -\nabla_y \log|x - y|$, so that, integrating by parts, we have

$$\lim_{R\to+\infty} w_R(Rx) = \frac{1}{|Q|} \int_{B_\alpha} \operatorname{div}\left(f(\alpha^2) - f(|y|^2)y\right) \log|x - y| dy.$$

This limit is a radially symmetric function, which solves the partial differential equation $\Delta u = \frac{2\pi}{|Q|} \operatorname{div}\left(f(\alpha^2) - f(|y|^2)y\right)$ in B_α, $\Delta u = 0$ elsewhere. For any primitive

F of f, the function $\frac{1}{2}\left(f(\alpha^2)|y|^2 - F(|y|^2)\right)\mathbf{1}_{B_\alpha}(y) + \frac{1}{2}\left(\alpha^2 f(\alpha^2) - F(\alpha^2)\right)\mathbf{1}_{B_\alpha^c}(y)$ is such a solution, so we have (5.47) almost everywhere. In addition, the above proof allows us to bound $2w_R(Rx) + (1/(v_{\alpha,R}^2) - 1)R^2|x|^2$ by $C - \frac{f(\alpha^2)}{4}|x|^2$ for some constant C independent of R and x, which allows us to apply the dominated convergence theorem.

We now prove (5.46). We fix $n = 1$, the general proof following exactly the same pattern. It is sufficient to show that the following convergence holds for any measurable bounded set D:

$$\int_D e^{v_R(Rx)} \underset{R\to+\infty}{\longrightarrow} |D| \fint e^v. \tag{5.58}$$

Hence, we are going to prove that for any $a > 0$,

$$\left|e^{v_R(x)} - e^{v(x)}\right| \leq C\frac{1+\sqrt{|x|}}{R} \quad \text{for} \quad |x| \leq aR. \tag{5.59}$$

This, together with the fact that $e^{v(Rx)}$ converges in L^∞ weak-* to $\fint e^v$ (because e^v is continuous and periodic), will give (5.58). Let j_x be the point of ℓ that is the closest to x. As R goes to infinity, $|(\lambda_R(|j_x|) - 1)j_x| = O\left(\frac{1}{R}\right)$ uniformly with respect to x since $|x| \leq aR$. Hence, for R large enough, $\lambda_R(|j_x|)j_x$ is the closest to x among all $\lambda_R(|j|)j$, $j \in \ell$. Hence, for $j \in \ell \setminus \{j_x\}$, we have, for some $\varepsilon > 0$,

$$\forall y \in Q, \quad |x - j - y| \geq \varepsilon \quad \text{and} \quad |x - j - \lambda_R(|j|)(j + y)| \geq \varepsilon. \tag{5.60}$$

We then isolate j_x in the sum defining v_R, and write

$$\left|e^{v_R(x)} - e^{v(x)}\right| \leq \left|e^{g_R(x-\lambda_R(|j_x|)j_x)} - e^{g_R(x-j_x)}\right| e^{\sum_{j\neq j_x} g_R(x-\lambda_R(|j|)j)} \tag{5.61}$$

$$+ e^{g_R(x-j_x)}\left|e^{\sum_{j\neq j_x} g_R(x-\lambda_R(|j|)j)} - e^{\sum_{j\neq j_x} g_R(x-j)}\right|, \tag{5.62}$$

where $g_R(z) = g\left(\frac{z}{v_{\alpha,R}}\right)$. We first bound (5.61). For this purpose, we point out that, according to Lemma 5.6, one can find a constant C_ε such that

$$\forall z \in B_\varepsilon^c, \quad |g(z)| \leq \frac{C_\varepsilon}{|z|^3}. \tag{5.63}$$

Hence, the sum appearing in (5.61) may be bounded as follows:

$$\sum_{j\neq j_x} |g_R(x - \lambda_R(|j|)j)| \leq \sum_{j\neq j_x} \frac{C_\varepsilon {v_{\alpha,R}}^3}{|x - \lambda_R(|j|)j|^3},$$

which is bounded independently of R. Moreover, we have

$$
\begin{aligned}
\left| e^{g_R(x-\lambda_R(|j_x|)j_x)} - e^{g_R(x-j_x)} \right| &\le C \, ||x - \lambda_R(|j_x|)j_x| - |x - j_x|| \\
&+ C \left| \int_Q \left(\log \frac{|x-y-j_x|}{|x-y-\lambda_R(|j_x|)j_x|} \right) dy \right| \\
&\le C \, |(1 - \lambda_R(|j_x|)j_x| \\
&+ C \int_Q \left(\frac{1}{|x-y-j_x|} + \frac{1}{|x-y-\lambda_R(|j_x|)j_x|} \right) |1 - \lambda_R(|j_x|)||j_x| dy \\
&\le C \frac{|x|}{R^2} + C \frac{|x|}{R^2} \int_{B_3} \frac{dy}{|y|} \le C \frac{|x|}{R^2}.
\end{aligned}
$$

Hence, the left-hand side of (5.61) is bounded by $C\frac{|x|}{R^2}$. Next, we deal with (5.62). Since g is bounded from above, it is sufficient to show the following:

$$
\left| \sum_{j \neq j_x} g_R(x - \lambda_R(|j|)j) - \sum_{j \neq j_x} g_R(x - j) \right| \le C \frac{1 + \sqrt{|x|}}{R}. \tag{5.64}
$$

In order to prove (5.64), we define $A > 0$ depending on R and x, to be fixed later on, and distinguish in the above sum between terms for which $|j - j_x| \le A$ and those for which $|j - j_x| > A$. We have

$$
\begin{aligned}
\sum_{0<|j-j_x|\le A} & |g_R(x - \lambda_R(|j|)j) - g_R(x - j)| \\
&\le \|\nabla g\|_{L^\infty(B_\varepsilon^c)} \sum_{0<|j-j_x|\le A} |j||\lambda_R(|j|) - 1| \\
&\le \frac{C}{R^2} \sum_{0<|j-j_x|\le A} |j| \le \frac{C}{R^2} A^2(|x| + A).
\end{aligned}
$$

We have used here the fact that g is Lipschitz continuous in B_ε^c. Considering the case $|j - j_x| > A$, we have, using (5.63),

$$
\sum_{A>|j-j_x|} |g_R(x - \lambda_R(|j|)j) - g_R(x - j)| \le \sum_{A>|j-j_x|} \frac{C}{|x-j|^3} \le \frac{C}{A}.
$$

We thus may bound the left-hand side of (5.64) by $\frac{C}{A} + \frac{CA^2|x|}{R^2} + \frac{CA^3}{R^2}$. Choosing $A = \frac{\sqrt{R}}{1+|x|^{\frac{1}{4}}}$, we thus obtain (5.64), thereby concluding the proof of (5.59).

5.4 Infinite number of zeros

In this section, we prove Theorem 5.4.

1. The proof first requires another formulation of (5.27). The projector Π has many properties [110, 9]; in particular, one can check, using an integration by parts

in the expression of Π, that $\Pi(|z|^2 f) = (z/\Omega)\partial_z f + f$. As for the middle term in the equation, one can compute that

$$\Pi\left(e^{-\Omega|z|^2}|f|^2 f\right) = \Pi\left(e^{-\Omega|z|^2}|f|^2\right)\Pi f$$

$$= \Pi(\overline{f}(z))\Pi(e^{-\Omega|z|^2} f^2) = \bar{f}\left(\frac{1}{\Omega}\partial_z\right)\Pi(e^{-\Omega|z|^2} f^2).$$

A simple change of variable yields

$$\Pi\left(e^{-\Omega|z|^2} f^2\right)(z) = \frac{\Omega}{\pi}\int e^{-\Omega z\overline{z'}-2\Omega|z'|^2} f^2(z')d^2r'$$
$$= \frac{1}{2}\Pi\left(f^2\left(\frac{\cdot}{\sqrt{2}}\right)\right)\left(\frac{z}{\sqrt{2}}\right) = \frac{1}{2}f^2\left(\frac{z}{2}\right).$$

Thus, we obtain the following simplification of (5.27):

$$(1-\Omega^2)z\partial_z f + Na\bar{f}\left(\frac{1}{\Omega}\partial_z\right)[f^2\left(\frac{z}{2}\right)] - (2\mu - 1 + \Omega^2)f = 0. \tag{5.65}$$

2. Now we assume that f is a polynomial of degree n and a solution of (5.65). We are going to show that there is a contradiction due to the term of highest degree in the equation. Indeed, if f is a polynomial of degree n, then $(\partial_z)^k[f^2(z/2)]$ is of degree $2n-k$. But (5.65) implies that $\bar{f}(\partial_z)[f^2(z/2)]$ is of degree n; hence f must be equal to cz^n. This is indeed a solution of (5.65) if $n(1-\Omega^2)+Na|c|^2(2n)!/(2^{2n}n!\Omega^n)-2\mu+1-\Omega^2 = 0$. Using that $\int |f|^2 e^{-\Omega|z|^2} = 1$, we find that $|c|^2\pi\Omega^{-n+1}n! = 1$. The Stirling formula provides the existence of a constant c_0 such that

$$n(1-\Omega^2) + \frac{c_0 Na\Omega}{2\pi\sqrt{n}} \le 2\mu + 1 - \Omega^2.$$

For the minimizer, μ is bounded by twice the energy, thus by $C\sqrt{1-\Omega}$, so that if Ω is close to 1, this implies that

$$2n\sqrt{1-\Omega} + \frac{c_0 Na\Omega}{2\pi\sqrt{n(1-\Omega)}} \le c_1,$$

and no n can satisfy this last identity. The minimizer is not a polynomial.

3. If f is a holomorphic function with a finite number of zeros, then there exist a polynomial $P(z)$ and a holomorphic function $\phi(z)$ such that $f = Pe^{\phi}$. The fact that $f \in \mathcal{F}$ provides a decreasing property on f [44], which implies that Re $(\phi(z)) \le \Omega|z|^2/2$. A classical result on holomorphic functions then yields that ϕ is a polynomial of degree at most two, and $f(z) = P(z)e^{\alpha+\beta z+\gamma z^2}$ with $|\gamma| \le \Omega/2$. A similar argument to that of case 2, but with more involved computations, provides a contradiction with the degree of the polynomial P if Ω is close to 1. We conclude that f has an infinite number of zeros. The detailed proof is given in [9].

5.5 Other trapping potentials

In the previous sections, we have studied a harmonic confinement, which is the case of most current experiments. The results of [7, 8] apply a more general trapping potential, where in (5.2), $(1-\Omega^2)r^2/2$ is replaced by $(1-\Omega^2)r^2/2 + W(r)$, and perform a similar analysis. Then, the limiting distribution replacing the inverted parabola should be

$$|\psi|^2 = \left(\frac{\mu - (1-\Omega)r^2 - W(r)}{Nab}\right)_+, \tag{5.66}$$

where μ is such that $\int |\psi|^2 = 1$. There are two necessary conditions for applying our previous analysis: we need a small parameter (replacing $1-\Omega$) such that $E_{\mathrm{LLL}}(\psi) - \Omega$ is small and the extent of the condensate (where $|\psi|^2$ is non zero) is large. The first condition is required so that the lowest Landau level is indeed a good approximation, and the second to apply the double scale convergence.

In recent experiments [40, 150], $W(r) = kr^4/4$. One can check that if $\Omega > \Omega_c = 1 + \sqrt{\Delta}$, where $\Delta = (3k^2 Nab/8\pi)^{2/3}$, then the limiting distribution (5.66) has its support in an annulus of inner and outer radii $R_\pm = 2(\Omega - 1 \pm \sqrt{\Delta})/k$. An interesting regime to study is that of k small and $\Omega - 1 = \alpha k^{2/3}$, with α such that $\Omega > \Omega_c$. Then the large scaling parameter replacing R is $k^{-1/6}$, which is the order of magnitude of $R_\pm$. The vortex lattice is located in the annulus (R_-, R_+) and is distorted towards the inner and outer edges, the inner disk corresponding to a giant vortex.

This approach does not allow us to study the case when Ω is large and the annulus gets thin [62], since in that case, we are no longer in the setting to apply double-scale convergence: there are few circles of vortices in the condensate. At even larger Ω, there is a phase transition in which the circle of vortices disappears into the giant vortex.

5.6 Open questions

5.6.1 Lower bound and Γ convergence

Our results deal with an upper bound for the energy. A natural question is to get also the lower bound and prove a Γ convergence result.

Open Problem 5.1 *Prove that if $\|\psi\|_{L^2} = 1$, and $\psi e^{\Omega|z|^2/2}$ is a holomorphic function, then*

$$\liminf_{\Omega\to 1} E_{\mathrm{LLL}}(\psi) \geq \frac{2\sqrt{2}}{3}\sqrt{\frac{Nab}{\pi}}(1-\Omega). \tag{5.67}$$

The present gap between the lower bound (5.15) and the upper bound (5.5) lies in the coefficient b. We believe that an optimal lower bound should match the upper bound and that the limiting inverted parabola should have a radius given by (5.19) instead of (5.14), that is, the optimal inverted parabola should have the coefficient b: reproducing an inverted parabola profile in the space (5.4) requires many vortices and thus creates a contribution in the energy through b. We have proved that the minimizer has an infinite number of zeros, but getting that these zeros are located on an almost regular lattice seems very difficult and is probably related to similar difficulties in crystallization and sphere packing problems.

Several paths can be sought. One of them could be to look for a homogenization expansion of the wave function. The main issue is the scale at which we expect the convergence, since there are two scales in the problem, the one of the lattice (order 1) and the other of the inverted parabola. If we use the scale of the lattice, we expect the following:

Open Problem 5.2 *Assume that ψ_Ω is a minimizer of (5.1) among functions $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ such that $f \in \mathcal{F}$. Fix a compact K and prove that in K, as Ω tends to 1, ψ_Ω converges to $\eta(z)$ given by (5.12).*

If we rescale the problem on the inverted parabola profile we conjecture this:

Open Problem 5.3 *Assume that ψ_Ω is a minimizer of (5.1) among functions $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ such that $f \in \mathcal{F}$. Prove that as Ω tends to 1, if R is given by (5.28), then $\psi_\Omega(Rz)$ converges to $\eta(z/R)p(Rz)$ in some weak sense, where η is given by (5.12) and p by (5.28).*

Another direction is to use the structure of the Fock–Bargmann space and the calculus with pseudo-differential operators. For instance, one could hope to prove that close to the test function $\Pi(p(z)\eta(z))e^{-\Omega|z|^2/2}$, there is a critical point of the energy. This would require a precise study of the Hessian about this test function.

5.6.2 Restriction to the LLL

Another issue is to check that the lower bound of the energy restricted to the LLL should provide the lower bound for the full energy (5.2) or that the reduction to the LLL well describes the full minimization of the energy.

Open Problem 5.4 *Prove that if ψ is a minimizer of (5.2) with $\|\psi\|_{L^2} = 1$, then ψ and $E(\psi) - \Omega$ are respectively close to ψ_Ω and $E_{\mathrm{LLL}}(\psi_\Omega)$, where ψ_Ω is a minimizer of (5.1) among functions $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ such that $f \in \mathcal{F}$.*

More precisely, if ψ is a minimizer of (5.2), we can project it on the LLL and its orthogonal through $\psi = \psi_{\mathrm{LLL}} + \psi_\perp$. The upper bound and the properties of the operator (5.3) imply that $\|\psi_{\mathrm{LLL}}\|_{L^2}$ is close to 1 and $\|\psi_\perp\|_{L^2}$ is small like $\sqrt{1-\Omega}$. This, and elliptic estimates, allow us [6] to prove that the energy of ψ decouples into $\Omega + E_{\mathrm{LLL}}(\psi_{\mathrm{LLL}})$ plus a lower order term, and eventually that $E(\psi) - \Omega$ is of the same order as $E_{\mathrm{LLL}}(\psi_\Omega)$, where ψ_Ω is a minimizer of (5.1) among functions $\psi(z) = f(z)e^{-\Omega|z|^2/2}$ such that $f \in \mathcal{F}$. We do not know how to prove that ψ and ψ_Ω are close in L^∞ norm, or equivalently that ψ_{LLL} and ψ_Ω are close.

5.6.3 Reduction to a two-dimensional problem

A related question is to prove that the reduction from a three-dimensional to a two-dimensional problem is justified.

Open Problem 5.5 *If Ω is close to 1 and ψ is a minimizer of the full three-dimensional Gross–Pitaevskii energy*

$$E_{3D}(\psi) = \int_{\mathbf{R}^3} \frac{1}{2}|\nabla\psi - i\mathbf{\Omega} \times \mathbf{r}\psi|^2 + \frac{1}{2}(1-\Omega^2)(x_1^2 + x_2^2)|\psi|^2 + \frac{1}{2}x_3^2|\psi|^2 + \frac{1}{2}Ng|\psi|^4, \tag{5.68}$$

with $\|\psi\|_{L^2} = 1$, then ψ is close to $\psi_{2d}(x_1, x_2)\xi(x_3)$, where ξ is a Gaussian in the x_3 direction and ψ_{2d} minimizes (5.2), with $a = g\int_{\mathbf{R}} \xi^4(x)\,dx$.

Indeed in the fast rotation regime, the effective trapping frequencies in the x_1 and x_2 directions, $\sqrt{1-\Omega^2}$ are much smaller when Ω tends to 1, than the frequency in the x_3 direction which is fixed, so that the wave function is expected to be on its ground state in the x_3 direction, which is a Gaussian. What we can prove for the moment [6] is that the projection of a 3D minimizer onto higher excited states w.r.t. the x_3-variable is small and one can take the projection on the lowest excited state ξ, as a test function for the 2D functional. From this we get $E_{3D}(\psi) = \inf E(\psi) + o(\sqrt{1-\Omega})$, where $E(\psi)$ is given by (5.2), with $a = g\int_{\mathbf{R}} \xi^4(x)\,dx$. Recall that $E(\psi) = O(\sqrt{1-\Omega})$. The fact that the wave functions are close is still open.

5.6.4 Mean field model

Studying the energy (5.2), known as the mean field quantum Hall regime, is acceptable only if the number of vortices is much smaller than the number of atoms in the condensate, which is the case of the present experiments. Otherwise, one has to consider other models, as in [50, 148]. The reduction of the N-body Hamiltonian to the Gross–Pitaevskii energy is an open question for this fast-rotating regime. It has been derived only in the case of fixed rotation by Lieb and Seiringer [98].

6

Three-Dimensional Rotating Condensate

In this chapter, we are interested in a three–dimensional rotating condensate, in a setting similar to that of the experiments. In particular, we want to justify the observations of the bent vortices. Thus we want to study the shape of vortices in minimizers of the following energy:

$$E_\varepsilon(u) = \int_{\mathcal{D}} \left\{ \frac{1}{2}|\nabla u|^2 - \boldsymbol{\Omega}_\varepsilon \times \mathbf{r} \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2}(|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}))^2 \right\} dx\, dy\, dz, \tag{6.1}$$

where $\mathbf{r} = (x, y, z)$, $\boldsymbol{\Omega}_\varepsilon$ is parallel to the z axis, $\rho_{\mathrm{TF}}(\mathbf{r}) = \rho_0 - (x^2 + \alpha^2 y^2 + \beta^2 z^2)$, $\mathcal{D}$ is the ellipsoid $\{\rho_{\mathrm{TF}} > 0\} = \{x^2 + \alpha^2 y^2 + \beta^2 z^2 < \rho_0\}$, and ρ_0 is determined by

$$\int_{\mathcal{D}} \rho_{\mathrm{TF}}(\mathbf{r}) = 1, \tag{6.2}$$

which yields $\rho_0^{5/2} = 15\alpha\beta/8\pi$. If β is small, as in the experiments, this gives rise to an elongated domain $\mathcal{D}$ along the z direction.

We intend to find an asymptotic expansion of the energy in terms of ε and determine the critical velocity for the existence of a vortex. Our mathematical results mainly deal with the single-vortex solution and are aimed at proving the bending property. In this derivation, the vortex can be represented by an oriented curve, which is a Lipschitz function $\gamma : (0, 1) \to \mathcal{D}$. We are going to prove that if the minimizer has a single singularity line γ, then the energy decouples into a contribution from the density profile η_ε (which is the minimizer of E_ε among vortex-free functions) and a contribution from the vortex line γ, that is, as ε tends to 0,

$$\frac{E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon)}{\pi |\log \varepsilon|} \sim E[\gamma], \tag{6.3}$$

where

$$E[\gamma] = \int_\gamma \rho_{\mathrm{TF}}\, dl - \bar{\Omega} \int_\gamma \rho_{\mathrm{TF}}^2\, dz, \quad \text{with} \quad \bar{\Omega} = \frac{\Omega_\varepsilon}{(1 + \alpha^2)|\log \varepsilon|}. \tag{6.4}$$

The energy $E[\gamma]$ reflects the competition between the vortex energy due to its length (first term) and the rotation term. Note that the rotation term is an oriented integral (dz not dl, which comes from the product $\mathbf{\Omega}_\varepsilon \cdot \mathbf{dl}$), which tends to force the vortex to be parallel to the z axis, while the other term wants to minimize the length. This is why, according to the geometry of the trap, the shape of the vortex varies. The study of the minimizer of $E[\gamma]$ will allow us to justify the bending property. This expansion (6.3) is analogous to the case of a single vortex in a 2D condensate, except that now the weight ρ_{TF} is integrated along the singularity line, instead of being evaluated at the vortex point p. The main difference with the 2D analysis is that there is no such lower bound as Proposition 3.10 to characterize the vortex tubes in terms of energy. This is at the origin of the less-precise results in 3D.

In the first section, we will present numerical simulations illustrating the different patterns for vortices. Then, in the next section, we will explain the formal asymptotic expansion for the energy, derived in [16]. In Section 3, we will describe the Γ convergence results of R. Jerrard [85], which prove that the reduced energy $E[\gamma]$ is indeed a Γ limit of $(E(u_\varepsilon) - E(u_\varepsilon)/)/\pi|\log\varepsilon|$. This uses the splitting of the energy into the energy of η_ε and an energy of $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$. Vortices are identified through the study of the Jacobians of v_ε, namely

$$Jv = \sum_{j<k} v_{x_j} \wedge v_{x_k}. \tag{6.5}$$

The convergence of Jacobians to a limiting current is proved and allows us to get regularity on the limiting singularity line γ. Finally, in Section 4, we will study the properties of the reduced energy $E[\gamma]$ and in particular get that the vortex line is indeed bending. This is based on [14, 15].

The main results can be summarized in the following theorem, which characterizes the critical velocity below which the minimizer is vortex-free:

Theorem 6.1. *Let* $\bar\Omega = \lim_{\varepsilon\to 0} \Omega_\varepsilon/(1+\alpha^2)|\log\varepsilon|$ *and*

$$\bar\Omega_1 = \inf\{\bar\Omega,\ \exists\gamma \text{ with } E[\gamma] < 0\}.$$

Then $1 < \bar\Omega_1\rho_0 < 5/4$. *For* $\bar\Omega < \bar\Omega_1$, *the global minimizer* u_ε *is asymptotically vortex-free in* $\mathcal{D}$, *that is, the Jacobian of* $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$ *tends to* 0. *For* $\bar\Omega > \bar\Omega_1$, *the minimizer has vortices. The straight vortex does not minimize the reduced energy* $E[\gamma]$ *if* $\beta < \sqrt{2/13}$.

6.1 Numerical simulations

In order to illustrate better our analytical results, we first present numerical simulations [10] of the different types of vortex patterns that we will study in the next section. Numerical simulations of solutions of the full Gross–Pitaevskii equations have been performed by I. Danaila [10]. Critical points of the original energy $E_\varepsilon(u)$ are computed by solving the imaginary time propagation of the corresponding equation:

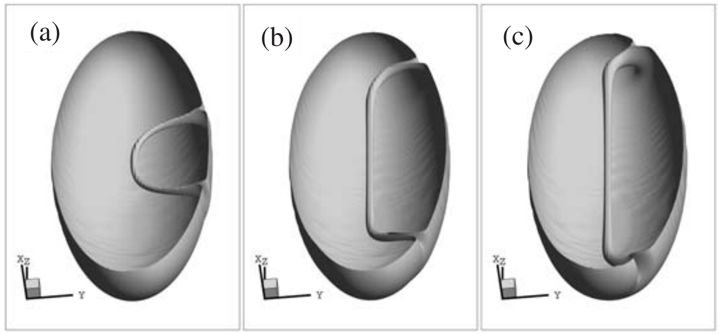


Fig. 6.1. Single U-vortex configurations for $\varepsilon\Omega_\varepsilon = 0.42$ (a), 0.58 (b), 0.78 (c).

$$\frac{\partial u}{\partial t} - \frac{1}{2}\nabla^2 u + i(\mathbf{\Omega} \times \mathbf{r}).\nabla u = \frac{1}{2\varepsilon^2} u(\rho_{\mathrm{TF}} - |u|^2), \tag{6.6}$$

with $u = 0$ on $\partial\mathcal{D}$. A hybrid three-step Runge–Kutta–Crank–Nicolson scheme is used to march in time. Various initial data, with or without vortices, are taken.

Three different types of single-vortex configurations are observed, as shown in Figure 1.5 in Chapter 1: planar U vortices, planar S vortices, and nonplanar S vortices. The U vortices are the global minimizers of the energy. The S configurations were observed experimentally recently [132] and are only local minimizers of the energy.

As will be explained in our analytical approach below, the U vortex is a planar vortex formed of two parts: the central part is almost a straight line that is very close to the z axis, and the outer part reaches the boundary of the condensate perpendicularly. When Ω increases, the central straight part gets longer, as illustrated in Figure 6.1. The U vortex lies either in the xz or yz plane. Starting with an initial condition that is not in one of these planes yields a final state in the yz plane, which is the plane closest to the z axis (we have taken α slightly bigger than 1). The U vortices exist only for Ω_ε bigger than a critical value Ω_0. It is interesting to note that at Ω_0, the energy of the U vortex is bigger than the energy of the vortex-free solution; Ω_0 is very close to the angular velocity Ω_1 for which the energy of the vortex-free solution is equal to the energy of the U vortex, yet slightly smaller. The angular momentum $L_z = \int_{\mathcal{D}} e_z \times r \cdot (iu, \nabla u)$ of the U vortex for $\Omega_\varepsilon = \Omega_0$ does not go to 0. This suggests that in fact there could be another U solution for $\Omega_\varepsilon > \Omega_0$. Using an ansatz, another type of U solution is obtained in [113], which is a saddle point of the energy: it is away from the axis and has lower angular momentum. We will discuss this issue below.

Motivated by the experiments of [132], we compute critical points of the energy, which are S configurations (see Figure 1.5). The planar S looks like a U, with the half-part in the plane $z < 0$ rotated with respect to the z axis by 180 degrees. But the analytical study will show us that the U vortex remains at finite distance (though very small) from the axis, while the S vortex goes through the origin. The difference in energy (and angular momentum) between U and S vortices is very small but S

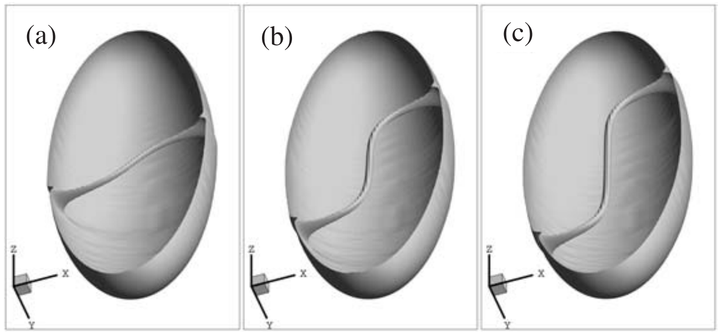


Fig. 6.2. Single S vortex configuration for $\varepsilon\Omega_\varepsilon = 0.38$ (a), 0.44 (b), 0.48 (c).

vortices are critical points of $E[\gamma]$ for any Ω, never minimizers of $E[\gamma]$. As already mentioned for the U vortex, stable planar S configurations lie either in the xz or yz plane. The nonplanar S are such that the projections of the branches on the xy plane are orthogonal, i.e., the rotation of the branches is of 90 degrees. We could check that nonplanar S configurations with an angle between the branches different from 90 degrees do not exist. The S vortices exist for all values of Ω_ε, while the U exist only for $\Omega_\varepsilon > \Omega_0$. When Ω_ε decreases, the extension of the S along the z axis goes downwards, as shown in Figure 6.2, and the vortex tends to the horizontal axis. Note that a vortex along the horizontal axis has $L_z = 0$, but a positive energy. On the other side, when Ω_ε increases, the S gets straighter and it tends to the vertical axis.

Numerical simulations in [10] also include the case of several vortices, which we do not reproduce here.

6.2 Formal derivation of the reduced energy $E[\gamma]$

Before presenting the rigorous Γ convergence result in the next section, we want to indicate formally the origin of the terms in the expansion of the energy. The main result of this section is the exact decoupling of the energy E_ε into three terms in (6.12): a part coming from the profile of the solution without vortices, a vortex contribution, and a term due to rotation. Then, each of these terms is formally evaluated in (6.21) and (6.22), but this will be made rigorous the Section 3. The analysis described in this section relies on [16].

6.2.1 The solution without vortices

Firstly, we are interested in the profile of solutions, so we will study solutions without vortices. Thus we consider functions of the form $\eta = f e^{iS}$, where f is real and does not vanish in the interior of $\mathcal{D}$. We first minimize E_ε over such functions. Then η_ε is a solution of

$$\nabla^2\eta_\varepsilon - 2i(\mathbf{\Omega}\times\mathbf{r}).\nabla\eta_\varepsilon + \frac{1}{2\varepsilon^2}\eta_\varepsilon(\rho_{\mathrm{TF}} - |\eta_\varepsilon|^2) = 0. \tag{6.7}$$

If the cross section is a disc, the phase is zero. When ε is small, since the ellipticity of the cross section is small, the zero-order approximation of f_ε^2 is ρ_{TF}. As for the phase, when the cross section is not a disc, its behavior is given by the continuity equation div $(f_\varepsilon^2(\nabla S_\varepsilon - \mathbf{\Omega}\times\mathbf{r})) = 0$. This implies that there exists $\mathbf{\Xi}_\varepsilon$ such that

$$f_\varepsilon^2(\nabla S_\varepsilon - \mathbf{\Omega}\times\mathbf{r}) = \Omega \text{ curl } \ \mathbf{\Xi}_\varepsilon. \tag{6.8}$$

One can think of $\mathbf{\Xi}_\varepsilon$ as the equivalent of a stream function in the case of fluid vortices. Dividing (6.8) by f_ε^2 and taking the curl, we find that $\mathbf{\Xi}_\varepsilon$ is the solution of

$$\text{curl }\left(\frac{1}{f_\varepsilon^2}\text{curl } \mathbf{\Xi}_\varepsilon\right) = -2 \text{ in } \mathcal{D}, \quad \mathbf{\Xi}_\varepsilon = 0 \text{ on } \partial\mathcal{D}. \tag{6.9}$$

When ε is small, the function $\mathbf{\Xi}_\varepsilon$ is well approximated by the solution $\mathbf{\Xi}$ of

$$\text{curl }\left(\frac{1}{\rho_{\mathrm{TF}}}\text{curl } \mathbf{\Xi}\right) = -2 \text{ in } \mathcal{D}, \quad \mathbf{\Xi} = 0 \text{ on } \partial\mathcal{D}. \tag{6.10}$$

One can easily get that $\mathbf{\Xi}(x,y) = -\rho_{\mathrm{TF}}^2(x,y)/(2+2\alpha^2)\mathbf{e}_z$. This exact expression is due to the harmonic potential: any other trapping potential does not allow an exact computation. Using (6.8), we can define S_0, the limit of S_ε, to be the solution of $\rho_{\mathrm{TF}}(\nabla S_0 - \mathbf{\Omega}\times\mathbf{r}) = \Omega$ curl $\mathbf{\Xi}$ with zero value at the origin. We have $S_0 = C\Omega xy$ with $C = (\alpha^2-1)/(\alpha^2+1)$. We see that S_0 vanishes when $\alpha = 1$, that is, when the cross section is a disc. The function $\eta_\varepsilon = f_\varepsilon e^{iS_\varepsilon}$ that we have studied gives the profile of any solution. It will allow us to compute the energy of all solutions.

6.2.2 Decoupling the energy

Let $\eta_\varepsilon = f_\varepsilon e^{iS_\varepsilon}$ be the vortex-free minimizer of E_ε discussed above. Let u_ε be a configuration that minimizes E_ε and let $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$. As in the previous chapter, we use the trick introduced in [97] to decouple the energy: since η_ε satisfies the Gross–Pitaevskii equation (6.7), we multiply it by $\eta_\varepsilon(1-|v_\varepsilon|^2)$ to get

$$\begin{aligned}\int_{\mathcal{D}}(|v_\varepsilon|^2-1)(-\frac{1}{2}\Delta f_\varepsilon^2 - \frac{1}{\varepsilon^2}f_\varepsilon^2(\rho_{\mathrm{TF}} - f_\varepsilon^2)\\ + |\nabla f_\varepsilon e^{iS_\varepsilon}|^2 - 2f_\varepsilon^2(\nabla S_\varepsilon\cdot\mathbf{\Omega}_\varepsilon\times\mathbf{r})) = 0.\end{aligned} \tag{6.11}$$

This leads to the following exact decoupling of the energy $E_\varepsilon(u_\varepsilon)$:

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + G_{\eta_\varepsilon}(v_\varepsilon) + I_{\eta_\varepsilon}(v_\varepsilon), \tag{6.12}$$

where

$$G_{\eta_\varepsilon}(v_\varepsilon) = \int_{\mathcal{D}} \frac{1}{2}|\eta_\varepsilon|^2|\nabla v_\varepsilon|^2 + \frac{|\eta_\varepsilon|^4}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2 \tag{6.13}$$

is the energy of vortices and

$$I_{\eta_\varepsilon}(v_\varepsilon) = \int_{\mathcal{D}} |\eta_\varepsilon|^2(\nabla S_\varepsilon - \mathbf{\Omega}_\varepsilon \times \mathbf{r}) \cdot (iv_\varepsilon, \nabla v_\varepsilon) \tag{6.14}$$

is the angular momentum of vortices. The first term $E_\varepsilon(\eta_\varepsilon)$ is independent of the solution u_ε, so we have to compute the next two and find for which configuration u_ε the minimum is achieved. We use that at zero order $|\eta_\varepsilon|^2 = f_\varepsilon^2$ is approximated by ρ_{TF} when ε is small, so that we can approximate G_{η_ε} by $G_{\sqrt{\rho_{\mathrm{TF}}}} = G_\varepsilon$ and I_{η_ε} by $I_{\sqrt{\rho_{\mathrm{TF}}}} = I_\varepsilon$.

Assuming that the solution u_ε has a vortex line along γ, that is, u_ε vanishes along γ with a winding number equal to 1, our aim is to estimate the energy of u_ε depending on γ. Our approximations rely on the fact that the ellipticity of the cross section is weak and that ε is sufficiently small. We refer to [16] for details.

6.2.3 Estimate of $G_\varepsilon(v_\varepsilon)$

We want to estimate

$$G_\varepsilon(v_\varepsilon) = \int_{\mathcal{D}} \frac{1}{2}\rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

The mathematical techniques to approximate G_ε are similar to those used in the previous chapter, inspired by [32] in dimension 2 and extended by [131] in dimension 3.

We define

$$T_{\lambda\varepsilon} = \{x \in \mathcal{D} \text{ s.t. } \mathrm{dist}(x,\gamma) \le \lambda\varepsilon\}, \tag{6.15}$$

and assume that $\lambda\varepsilon$ is small, λ being our matching parameter to be fixed later on. Then we split G_ε into two integrals: one in $T_{\lambda\varepsilon}$, the energy of the vortex core, and the other in $\mathcal{D} \setminus T_{\lambda\varepsilon}$, the energy away from the vortex core.

At each point $\gamma(t)$ of γ, we define $\Pi^{-1}(\gamma(t))$ to be the plane orthogonal to γ at $\gamma(t)$. Since $\lambda\varepsilon$ is small, we assume that ρ_{TF} is constant in $\Pi^{-1}(\gamma(t)) \cap T_{\lambda\varepsilon}$ and we define the value $\rho_t = \rho_{\mathrm{TF}}(\gamma(t))$. We want to compute

$$\int_{T_{\lambda\varepsilon}} \frac{1}{2}\rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1-|v_\varepsilon|^2)^2$$
$$\sim \int_\gamma \frac{\rho_t}{2} \int_{\Pi^{-1}(\gamma(t))\cap T_{\lambda\varepsilon}} |\nabla v_\varepsilon|^2 + \frac{\rho_t}{2\varepsilon^2}(1-|v_\varepsilon|^2)^2.$$

This computation is valid as long as $k\lambda\varepsilon$ is small, where k is the curvature of γ. The zero-order approximation of v_ε is given by $u_1(r\sqrt{\rho_t}/\varepsilon)$, where $u_1(r,\theta) = f_1(r)e^{i\theta}$ is the solution with a single zero at the origin of the cubic NLS equation

$$\Delta u + u(1 - |u|^2) = 0 \text{ in } \mathbf{R}^2.$$

Thus,

$$\begin{aligned}
\int_{\Pi^{-1}(\gamma(t))\cap T_{\lambda\varepsilon}} & |\nabla v_\varepsilon|^2 + \frac{\rho_t}{2\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \\
&\sim \int_{B_{\lambda\varepsilon}} \left|\nabla\Big(f_1(r\sqrt{\frac{\rho_{\mathrm{TF}}}{\varepsilon^2}})e^{i\theta}\Big)\right|^2 + \frac{\rho_t}{2\varepsilon^2}\Big(1 - f_1^2\Big(r\sqrt{\frac{\rho_{\mathrm{TF}}}{\varepsilon^2}}\Big)\Big)^2 \\
&= \int_{B_{\lambda\sqrt{\rho_t}}} |\nabla u_1|^2 + \frac{1}{2}(1 - |u_1|^2)^2 \\
&\sim c_* + 2\pi \log(\lambda\sqrt{\rho_t}),
\end{aligned} \tag{6.16}$$

where

$$c_* = \int_{\mathbf{R}^2} f_1'^2 + \frac{1}{2}(1 - f_1^2)^2 + \int_{\mathbf{R}^2\setminus B_1} \frac{f_1^2 - 1}{r^2} + \int_{B_1} \frac{f_1^2}{r^2}.$$

The last line of (6.16) would be an equality if the first two integrals in the expression of c_* were taken in $B_{\lambda\sqrt{\rho_t}}$ instead of $\mathbf{R}^2$. This approximation is correct if $\lambda\sqrt{\rho_t}$ is large (in fact bigger than 3 is enough).

The final estimate is

$$G_\varepsilon(v_\varepsilon)_{|T_{\lambda\varepsilon}} \sim \int_\gamma \rho_{\mathrm{TF}}(\frac{c_*}{2} + \pi \log(\lambda\sqrt{\rho_{\mathrm{TF}}}))dl. \tag{6.17}$$

We are going to estimate G_ε in $\mathcal{D}\setminus T_{\lambda\varepsilon}$. In this region $|v_\varepsilon| \approx 1$, and we have seen that $\lambda\sqrt{\rho_t}$ is large, so that only the kinetic energy of the phase has a contribution:

$$\int_{\mathcal{D}\setminus T_{\lambda\varepsilon}} \frac{1}{2}\rho_{\mathrm{TF}}|\nabla v_\varepsilon|^2 + \frac{\rho_{\mathrm{TF}}^2}{4\varepsilon^2}(1 - |v_\varepsilon|^2)^2 \sim \int_{\mathcal{D}\setminus T_{\lambda\varepsilon}} \frac{1}{2}\rho_{\mathrm{TF}}|\nabla\phi_\varepsilon|^2,$$

where ϕ_ε is the phase of v_ε. We let $\boldsymbol{\Psi}$ be a stream function that is div $\boldsymbol{\Psi} = 0$ and

$$\operatorname{curl} \boldsymbol{\Psi} = \rho_{\mathrm{TF}}\nabla\phi.$$

Then $\boldsymbol{\Psi}$ is the unique solution of

$$\operatorname{curl}\Big(\frac{1}{\rho_{\mathrm{TF}}}\operatorname{curl} \boldsymbol{\Psi}\Big) = 2\pi\boldsymbol{\delta}_\gamma,\ \boldsymbol{\Psi} = 0 \text{ on } \partial\mathcal{D}, \tag{6.18}$$

where $\boldsymbol{\delta}_\gamma$ is the vectorial Dirac measure along γ. Thus,

$$\int_{\mathcal{D}\setminus T_{\lambda\varepsilon}} \frac{1}{2}\rho_{\mathrm{TF}}|\nabla\phi_\varepsilon|^2 = \int_{\mathcal{D}\setminus T_{\lambda\varepsilon}} \frac{1}{2\rho_{\mathrm{TF}}}|\operatorname{curl} \boldsymbol{\Psi}|^2 = -\frac{1}{2}\int_{\partial T_{\lambda\varepsilon}} \boldsymbol{\Psi}\cdot\nabla\phi_\varepsilon \times \nu,$$

where ν is the outward unit normal to the tube $T_{\lambda\varepsilon}$. We will see that $\boldsymbol{\Psi}$ is almost constant at a distance $\lambda\varepsilon$ from γ and we call this value $\boldsymbol{\Psi}_{\lambda\varepsilon}(\gamma)$. Since the vortex line has a winding number 2π,

$$\int_{\mathcal{D}\setminus T_{\lambda\varepsilon}} \frac{1}{2}\rho_{\mathrm{TF}}|\nabla\phi_\varepsilon|^2 \sim \pi \int_\gamma \mathbf{\Psi}_{\lambda\varepsilon}(\gamma)\cdot dl.$$

We have to compute $\mathbf{\Psi}$ on $\partial T_{\lambda\varepsilon}$. The computation is inspired by the paper of Svidzinsky and Fetter [152]. Let $x_0 \in \gamma$. We set $\mathbf{e}_3 = \dot{\gamma}(x_0)$ and suppose $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ to be an orthogonal base in local coordinates. Then $\mathbf{\Psi}$ has coordinates ψ_i in $\mathbf{e}_i$ and the variations of ψ_3 are the only ones of influence in the equation for $\mathbf{\Psi}$, since we want to compute $\mathbf{\Psi}\cdot dl$. Let $\xi = \psi_3/\sqrt{\rho_{TF}}$. Then ξ satisfies

$$-\Delta\xi + \mu\xi = 2\pi\sqrt{\rho_{\mathrm{TF}}}\boldsymbol{\delta}_\gamma,$$

where

$$\mu = \sqrt{\rho_{\mathrm{TF}}}\Delta\frac{1}{\sqrt{\rho_{\mathrm{TF}}}} = \sqrt{\rho_{\mathrm{TF}}}\Delta_\perp\frac{1}{\sqrt{\rho_{\mathrm{TF}}}}.$$

Here $\Delta_\perp$ is the Laplacian in the plane perpendicular to $\mathbf{e}_3 = \dot{\gamma}(x_0)$. If the cross section of the condensate $\mathcal{D}$ is a disc, one can compute μ. We denote by θ the angle of $\mathbf{e}_3$, that is, $\mathbf{e}_3 = \cos\theta\mathbf{e}_r + \sin\theta\mathbf{e}_z$, and (r, z) are the coordinates of x_0 in the original frame. Then

$$\mu = \frac{(1+\sin^2\theta)+\beta^2\cos^2\theta}{\rho_{\mathrm{TF}}} + \frac{3(r\sin\theta - \beta^2 z\cos\theta)^2}{\rho_{\mathrm{TF}}^2}.$$

Note that $\mu > 0$. We locally approximate the curve γ near the point x_0 by the parabola $x = kz^2/2$, where k is the curvature of γ at x_0. Note that in our approximations, we are only taking into account the shape of γ close to x_0. The justification for this relies on the fact that μ is large enough. We rewrite (6.2.3) in local coordinates to get

$$-\Delta\left(e^{\frac{-kx_1}{2}}\xi\right) + \left(\left(\frac{k}{2}\right)^2 + \mu\right)\left(e^{\frac{-kx_1}{2}}\xi\right) = 2\pi\sqrt{\rho_{\mathrm{TF}}(x_0)}\delta_{\mathbf{e}_3}.$$

The solution of this equation is

$$\sqrt{\rho_{\mathrm{TF}}(x_0)}K_0\left(\sqrt{\mu + \frac{k^2}{4}}\mathrm{dist}(x,\gamma)\right),$$

where K_0 is a modified Bessel function. In particular, $K_0(x) \sim -\log(e^{C_0}x/2)$ for small x, where $C_0 \approx 0.577$ is the Euler constant. Hence, we deduce

$$\mathbf{\Psi}(x) \approx -\rho_{\mathrm{TF}}\log\left(\frac{e^{C_0}}{2}\sqrt{\mu+\frac{k^2}{4}}\mathrm{dist}(x,\gamma)\right)\dot{\gamma}.$$

Thus we conclude by the estimate for $G_\varepsilon(v_\varepsilon)$ in $\mathcal{D}\setminus T_{\lambda\varepsilon}$ that

$$G_\varepsilon(v_\varepsilon)_{|\mathcal{D}\setminus T_{\lambda\varepsilon}} \sim -\pi\int_\gamma \rho_{\mathrm{TF}}\log\left(\frac{e^{C_0}}{2}\sqrt{\mu+\frac{k^2}{4}}\lambda\varepsilon\right) dl. \tag{6.19}$$

Here we have used that $\lambda\varepsilon$ is sufficiently small. In the previous section, we needed $\lambda\sqrt{\rho_t}$ large. The existence of λ is justified by the fact that $\sqrt{\rho_{\mathrm{TF}}}/\varepsilon$ is much bigger than 1, except very close to the boundary. But in this region, the contribution of the energy is negligible.

We find that each vortex line γ provides a contribution

$$G_\varepsilon(v_\varepsilon) \sim \int_\gamma \rho_{\mathrm{TF}} \left(\frac{c_*}{2} + \pi \log\left(\frac{2}{\varepsilon e^{C_0}} \sqrt{\frac{\rho_{\mathrm{TF}}}{\mu + \frac{k^2}{4}}} \right) \right) dl, \tag{6.20}$$

which can be approximated by

$$G_\varepsilon(v_\varepsilon) \sim \pi |\log \varepsilon| \int_\gamma \rho_{\mathrm{TF}}\, dl. \tag{6.21}$$

6.2.4 Estimate of $I_\varepsilon(v_\varepsilon)$

The estimate for I_ε is very similar to that in the 2D case. Recall that the unique solution of (6.9) satisfies $\rho_{\mathrm{TF}}(\nabla S_\varepsilon - \mathbf{\Omega} \times \mathbf{r}) = \Omega \operatorname{curl} \mathbf{\Xi}_\varepsilon$. Hence we integrate by parts in the expression for $I_{\eta_\varepsilon}(v_\varepsilon)$ to get

$$I_{\eta_\varepsilon}(v_\varepsilon) = \Omega \int_{\mathcal{D}} \mathbf{\Xi}_\varepsilon \cdot \operatorname{curl}\,(i v_\varepsilon, \nabla v_\varepsilon).$$

Let ϕ_ε be the phase of v_ε. Since v_ε is tending to one everywhere except on the vortex line, then $(i v_\varepsilon, \nabla v_\varepsilon) \sim \nabla \phi_\varepsilon$; hence we can approximate curl $(i v_\varepsilon, \nabla v_\varepsilon)$ by $2\pi \boldsymbol{\delta}_\gamma$. We use the value of $\mathbf{\Xi}$ and the fact that $\dot\gamma(t) \cdot \mathbf{e}_z = dz$, to obtain

$$I_\varepsilon(v_\varepsilon) \sim -\frac{\Omega\pi}{(1+\alpha^2)} \int_\gamma \rho_{\mathrm{TF}}^2\, dz. \tag{6.22}$$

6.2.5 Final estimate for the energy

We use (6.12), (6.21), and (6.22) to derive the energy of a solution with a vortex line. The energy of any solution minus the energy of a solution without vortex is roughly the vortex contribution in the sense $(E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon))/|\log \varepsilon| \sim E[\gamma]$, where $E[\gamma]$ is given by (6.4).

Let us point out that Svidzinsky and Fetter [152] have studied the dynamics of a vortex line depending on its curvature. For a vortex velocity equal to 0, the equation obtained in [152] is the same as the equation corresponding to the minimum of our approximate energy, though the formulation in [152] was not derived from energy considerations. Following our work, Modugno et al. [113] have also derived an approximate expression for the energy. Note that the energy that we actually derive in [16] is slightly more involved than (6.4). In the regime of the experiments, it is reasonable to restrict to this expression (6.4), taking into account the fact that the vortex core is sufficiently small (it is of size ε in our units) and neglecting the interaction of the curve with itself.

When there are several vortices, the energy has an extra term due to the repulsion between the lines $I(\gamma_i, \gamma_k)$:

$$E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon) \approx \sum_i E[\gamma_i] + \sum_{i \neq k} I(\gamma_i, \gamma_k), \tag{6.23}$$

where

$$I(\gamma_i, \gamma_k) = \pi \int_{\gamma_i} \rho_{\mathrm{TF}} \log(\mathrm{dist}(x, \gamma_k))\, dl.$$

6.3 Γ convergence results

6.3.1 Main results

A rigorous mathematical derivation of $E[\gamma]$ using Γ-convergence has been performed in [85]. The proof uses the splitting of the energy (6.12) and the definition of $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, where η_ε is a minimizer over vortex-free solutions. Since u_ε and v_ε have the same phase singularities and since $|v_\varepsilon| \sim 1$, the asymptotic singularities of v_ε can be understood by finding the limits of the Jacobian (6.5) of v_ε. This is used to identify the limiting objects as currents and prove regularity results to deduce that they are Lipschitz curves representing the vortex filaments.

The precise statement of the theorem requires the introduction of some notation and tools. It is helpful to reformulate $E[\gamma]$ as a functional acting on currents: if γ is a parameterized curve, one can define a current T_γ corresponding to the integration along γ by

$$T_\gamma(\phi) = \int_0^1 \phi^i(\gamma(t))\dot{\gamma}^i(t)\, dt, \quad \text{for} \quad \phi = \phi^i dx^i \in C_c^\infty(\mathcal{D}; \Lambda^1\mathbf{R}^3), \tag{6.24}$$

where $\Lambda^k\mathbf{R}^n$ denotes the space of k-covectors in $\mathbf{R}^n$, with the basis $\{dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}\ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$. We define $\Lambda_k\mathbf{R}^n$ to be the space of k-vectors with the basis $\{e^{\alpha_1} \wedge \cdots \wedge e^{\alpha_k}\ 1 \leq \alpha_1 < \cdots < \alpha_k \leq n\}$. The dual pairing between vectors and covectors is denoted by $\langle ., . \rangle$.

More generally, a k-dimensional current T on an open set U is a bounded linear functional on the space of smooth k-forms with compact support in U. The boundary of a k-dimensional current is the $(k-1)$-dimensional current ∂T defined by $\partial T(\phi) = T(d\phi)$. A current T is said to have locally finite mass in $U \subset \mathbf{R}^n$ if it can be represented in the form

$$T(\phi) = \int_U \langle \phi, \overrightarrow{T} \rangle\, d\|T\| \quad \text{for} \quad \phi \in C_c^\infty(U; \Lambda^k\mathbf{R}^n), \tag{6.25}$$

where $\|T\|$ is a nonnegative Radon measure, locally finite in U, and $\overrightarrow{T}$ is a $\|T\|$ measurable function taking values in $\Lambda_k\mathbf{R}^n$, normalized by $|\overrightarrow{T}| = 1$ almost everywhere.

Using these definitions for currents, we now point out that the functional $E[\gamma]$ can be extended to 1-dimensional currents with locally finite mass such that

$$M_{\rho_{\mathrm{TF}}}(T) := \int \rho_{\mathrm{TF}} \|T\|(dx) = \sup \left\{ T(\phi),\ \phi \in C_c^\infty(\mathcal{D}; \Lambda^1 \mathbf{R}^3), \left\| \frac{\phi}{\rho_{\mathrm{TF}}} \right\|_\infty \leq 1 \right\} \tag{6.26}$$

is finite by

$$E[T] = \int \rho_{\mathrm{TF}} \|T\|(dx) - \bar{\Omega} T(\rho_{\mathrm{TF}}^2 dx^3), \tag{6.27}$$

where the first term is the weighted mass of the current. If γ is a parameterized Lipschitz curve, if we compare (6.4), (6.24), (6.27), we indeed have $E[\gamma] = E[T_\gamma]$. The currents that we will consider will be rectifiable 1-dimensional currents with locally finite mass. It is proved in [85] that they can be identified as a countable sum of oriented Lipschitz curves γ_i, that is, $T = \sum_i T_{\gamma_i}$.

Recall that the definition of the Jacobian of an H^1 function is given by (6.5). It is convenient to associate with Jv a 1-current defined by

$$\star Jv(\phi) = \int_{\mathcal{D}} \phi \wedge Jv, \quad \phi \in C_c^\infty(\mathcal{D}; \Lambda^1 \mathbf{R}^3). \tag{6.28}$$

We are now going to state the main result, with notation close to that of Section 6.2.1.

Theorem 6.2 (R. Jerrard, [85]). *Assume that $\Omega_\varepsilon/(1+\alpha^2)|\log \varepsilon|$ tends to $\overline{\Omega}$, and that u_ε is a sequence of minimizers of E_ε. Let f_ε be the minimizer in $H^1(\mathcal{D}; \mathbf{R})$ of*

$$F_\varepsilon(f) = \int_{\mathcal{D}} \frac{1}{2} |\nabla f|^2 + \frac{1}{4\varepsilon^2} (\rho_{\mathrm{TF}} - f^2)^2. \tag{6.29}$$

Let

$$\eta_\varepsilon = f_\varepsilon e^{i\Omega_\varepsilon S_0} \quad \text{with} \quad S_0 = \frac{\alpha^2 - 1}{\alpha^2 + 1} x_1 x_2. \tag{6.30}$$

Assume that $G_{\eta_\varepsilon}(v_\varepsilon) \leq C|\log \varepsilon|$, where $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$, and $G_{\eta_\varepsilon}(v_\varepsilon)$ is defined by (6.13). Then there exists a one-dimensional rectifiable integral current J with locally finite mass such that as ε tends to 0,

$$\frac{(E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon))}{|\log \varepsilon|} \quad \Gamma\text{-converges to} \quad E[J], \tag{6.31}$$

where $E[J]$ is defined by (6.27).

The precise statement of Γ-convergence will be given below in Theorem 6.5. The study of minimizers of $E[J]$ allows us to get more properties of J, as we will see in the next section. The detailed proof of this theorem is quite delicate and requires a topological framework for the study of currents due to the degeneracy of ρ_{TF} near $\partial\mathcal{D}$. We will try to explain the main ideas of the proof. A specific assumption that we want to point out is that $G_{\eta_\varepsilon}(v_\varepsilon) \leq K|\log \varepsilon|$. This does not come from the a priori estimates of the energy and implies in some sense that there is a bounded number of vortices in $\mathcal{D}$.

Remark 6.3. The definition of η_ε given by (6.30) is not the same as in the previous section. Indeed, instead of writing S_ε, we directly take the limit S_0 in the phase. This slightly complicates the computation of the splitting of the energy but avoids proving convergence results for S_ε. We have in particular the following property for S_0:

$$\rho_{\mathrm{TF}}(\nabla S_0 - \Omega \times r) = -\mathrm{curl}\left(\frac{\rho_{\mathrm{TF}}^2 e_z}{2(1+\alpha^2)}\right). \tag{6.32}$$

Let us mention several consequences of this theorem:

Theorem 6.4 (R. Jerrard, [85]). *Assume that* $\bar{\Omega} < \bar{\Omega}_1$ *defined in Theorem 6.1, and let* u_ε *be a minimizer of* E_ε*. Then for* $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$*, we have* $\star J v_\varepsilon \to 0$*, as* ε *tends to* 0*, that is,* u_ε *is asymptotically vortex-free.*

Another consequence of the Γ convergence properties proved in [85], which is an application of the Kohn–Sternberg scheme [94], states that if γ_0 is a local minimizer of $E[\gamma]$, then there exists a local minimizer of E_ε whose vorticity is close to γ_0, in the sense of the Jacobians of v_ε. The precise statement of this theorem [85] involves the introduction of a topology on currents to define local minimizers of $E[T]$, and in particular of seminorms, due to the degeneracy of ρ_{TF} close to the boundary. In fact, we will see in the next section that the problem of minimizing $E[T]$ can be reduced to a two-dimensional domain in the plane $x = 0$. There, the current can be expressed in terms of a BV function.

The limiting energy $E[J]$ is either positive or unbounded below, depending on the value of $\bar{\Omega}$: if γ minimizes $E[\gamma]$ then the curve γ taken k times has energy $kE[\gamma]$. But this does not reflect the finite ε behaviour. Indeed, the derivation of $E[J]$ erases lower-order terms in ε, taking into account the interactions. These terms can be neglected in studying suitable local minimizers. We refer to [85] for more details. A more careful analysis of the interaction terms would be needed for a complete analysis of the minimizers of E_ε.

6.3.2 Main ideas in the proof

Here is a more refined version of what we will prove, which provides a detailed formulation of the Γ convergence results. This requires a norm for the convergence of currents:

$$\|T\|_\varepsilon := \sup\left\{T(\phi),\ \phi \in C_c^\infty(\mathcal{D}; \Lambda^1\mathbf{R}^3),\ \text{s.t.}\ \left\|\frac{\phi}{f_\varepsilon^4}\right\|_\infty + \left\|\frac{\nabla\phi}{f_\varepsilon^2}\right\|_\infty \le 1\right\}, \tag{6.33}$$

where f_ε is defined in Theorem 6.2. The exact statement of the following theorem is given in [85] and requires the introduction of seminorms for currents, due to the degeneracy of ρ_{TF} close to the boundary. We have chosen here to present a simplified version of the result.

Theorem 6.5 (R. Jerrard, [85]). *Assume that* $\Omega_\varepsilon/(1+\alpha^2)|\log\varepsilon|$ *tends to* $\bar{\Omega}$ *as* ε *tends to* 0 *and that*

$$\frac{1}{|\log \varepsilon|} G_{\eta_\varepsilon}(v_\varepsilon) \leq C m_\varepsilon \quad \text{with } 1 \leq m_\varepsilon \leq |\log \varepsilon|, \tag{6.34}$$

where we use the notation of Theorem 6.2. There exists ε_0 such that for all $\varepsilon < \varepsilon_0$, there exists a current $\tilde{J}_\varepsilon v_\varepsilon$ close to $\star J v_\varepsilon$ such that

$$\partial(\tilde{J}_\varepsilon v_\varepsilon) = 0, \quad M_{\rho_{\mathrm{TF}}}\left(\frac{\tilde{J}_\varepsilon v_\varepsilon}{m_\varepsilon}\right) \leq C, \quad \|\tilde{J}_\varepsilon v_\varepsilon - \star J v_\varepsilon\|_\varepsilon \leq \varepsilon^\delta \tag{6.35}$$

for some δ. Moreover, $\star J v_\varepsilon / m_\varepsilon$ is precompact as a sequence of distributions, and if J is any limit of a convergent subsequence of $\star J v_\varepsilon / m_\varepsilon$, then $M_{\rho_{\mathrm{TF}}}(J)$ is finite and

$$\liminf_{\varepsilon \to 0} \frac{1}{m_\varepsilon |\log \varepsilon|}(E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon)) \geq E[J]. \tag{6.36}$$

In addition, if $m_\varepsilon = 1$, then $(1/\pi)J$ is a 1-rectifiable locally finite current with no boundary.

For any such J also satisfying that $M_{\rho_{\mathrm{TF}}}(J)$ is finite, there exists a sequence of functions u_ε such that $\star J v_\varepsilon \to J$ in the sense of currents and

$$\lim_{\varepsilon \to 0} \frac{1}{|\log \varepsilon|}(E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon)) = E[J]. \tag{6.37}$$

The current $\tilde{J}_\varepsilon v_\varepsilon$ is obtained from $J v_\varepsilon / m_\varepsilon$ by modifying it in two ways: $J v_\varepsilon$ is regularized by convolution with a smoothing kernel. This is necessary since $G_{\eta_\varepsilon}(v_\varepsilon)$ does not control $J v_\varepsilon$ in L^1 but controls the smoothed current. Then the regularized current is modified near the boundary since all estimates on the energy are for G_{η_ε}, and thus require lower bounds for f_ε, which are true only away from the boundary. The properties proved for the current $\tilde{J}_\varepsilon v_\varepsilon$ allow us to obtain properties for the limit J.

Theorem 6.2 is stated with $m_\varepsilon = 1$ for simplicity. We will see that if $m_\varepsilon = |\log \varepsilon|$, then in fact (6.34) is no assumption, because it directly comes from the energy estimates.

In the proof of Theorem 6.4, the difficulty is to get from the general estimate $m_\varepsilon = |\log \varepsilon|$ that in fact $m_\varepsilon = 1$. The rest is a consequence of (6.36), which is a negative quantity, and the definition of $\bar{\Omega}_1$, which implies that $J = 0$.

We will now present the proof of Theorem 6.5. Let us write the proof for $m_\varepsilon = 1$ ($m_\varepsilon = |\log \varepsilon|$ is useful for Theorem 6.4).

Basic estimates and splitting the energy

Let us define the energy density

$$g_\varepsilon(u) = \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(1 - |u|^2)^2. \tag{6.38}$$

Lemma 6.6. *Let u_ε be a minimizer of E_ε. Then*

$$E_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|, \quad \int_{\mathcal{D}} g_\varepsilon(u_\varepsilon) \le C|\log \varepsilon|^2. \tag{6.39}$$

Proof: The first inequality comes from the construction of a test function, equal to $\sqrt{\rho}_{\mathrm{TF}}$ in the bulk of the condensate and linear close to the boundary. We define $\xi(s) = \gamma(\rho_{\mathrm{TF}}(s))$, where

$$\gamma(s) = \begin{cases} \sqrt{s}, & \text{if } s > \varepsilon^{2/3}, \\ \frac{s}{\varepsilon^{1/3}}, & \text{if } s < \varepsilon^{2/3}. \end{cases}$$

Using the coarea formula, we obtain

$$\int_{\mathcal{D}} |\nabla \xi|^2 = \int \gamma'(\rho_{\mathrm{TF}}(r))^2 |\nabla \rho_{\mathrm{TF}}|^2 \le C \int_0^{\rho_0} \gamma'(s)^2 \, ds \le C|\log \varepsilon|.$$

For the other term,

$$\int_{\mathcal{D}} (\rho_{\mathrm{TF}} - \gamma(\rho_{\mathrm{TF}})^2)^2 \le \int_0^{\varepsilon^{2/3}} (s - \gamma(s)^2)^2 \, ds \le C\varepsilon^2.$$

Hence, the energy of this test function is bounded by $|\log \varepsilon|$.

The second inequality follows from the Cauchy–Schwarz inequality:

$$\begin{aligned} \left| \int_{\mathcal{D}} \mathbf{\Omega}_\varepsilon \times \mathbf{r} \cdot (iu, \nabla u) \right| &\le \int_{\mathcal{D}} \frac{1}{4} |\nabla u_\varepsilon|^2 + C\Omega_\varepsilon^2 |u_\varepsilon|^2 \\ &\le \frac{1}{2} \int_{\mathcal{D}} g_\varepsilon(u_\varepsilon) + C\Omega_\varepsilon^2 (1 + \varepsilon^2 \Omega_\varepsilon^2). \quad \square \end{aligned} \tag{6.40}$$

Lemma 6.7. *The splitting of the energy holds:*

$$E_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + G_{\eta_\varepsilon}(v_\varepsilon) + I^1_{\eta_\varepsilon}(v_\varepsilon) + I^2_{\eta_\varepsilon}(v_\varepsilon), \tag{6.41}$$

where $G_{\eta_\varepsilon}(v_\varepsilon)$ is defined by (6.13) and

$$I^1_{\eta_\varepsilon}(v_\varepsilon) = \Omega_\varepsilon \int_{\mathcal{D}} f_\varepsilon^2 (\nabla S_0 - e_z \times r)(iv_\varepsilon, \nabla v_\varepsilon), \tag{6.42}$$

$$I^2_{\eta_\varepsilon}(v_\varepsilon) = \frac{\Omega_\varepsilon^2}{2} \int_{\mathcal{D}} f_\varepsilon^2 (|v_\varepsilon|^2 - 1)(|\nabla S_0|^2 - e_z \times r \cdot \nabla S_0). \tag{6.43}$$

The proof is the same as in the previous section. It consists in substituting $u_\varepsilon = \eta_\varepsilon v_\varepsilon$ in the energy and multiplying the equation satisfied by f_ε by $(|v_\varepsilon|^2 - 1)$ to get the result.

Lemma 6.8. *Let f_ε be the minimizer of F_ε. Then there exists a constant C such that*

$$0 < \sqrt{\rho_{\mathrm{TF}}} - f_\varepsilon < C\varepsilon^{1/6} \quad \forall x \in \mathcal{D}. \tag{6.44}$$

Proof: Since f_ε is a minimizer of F_ε, it satisfies

$$-\Delta(f_\varepsilon)^2 + 2|\nabla f_\varepsilon|^2 + \frac{2}{\varepsilon^2}(f_\varepsilon^2 - \rho_{\mathrm{TF}}) f_\varepsilon = 0.$$

Since $\Delta\rho_{\mathrm{TF}} < 0$, we deduce that

$$\Delta(f_\varepsilon^2 - \rho_{\mathrm{TF}}) > \frac{2}{\varepsilon^2}(f_\varepsilon^2 - \rho_{\mathrm{TF}}) f_\varepsilon,$$

and the strong maximum principle implies that $f_\varepsilon^2 < \rho_{\mathrm{TF}}$ in $\mathcal{D}$. The second inequality is proved in [85] and relies on the Jensen inequality and the fact that f_ε is superharmonic. ◻

Jacobian estimates

In this section, we recall earlier estimates of [86] on Jacobians that show how the energy density $g_\varepsilon(v_\varepsilon)$ controls the Jacobian.

Lemma 6.9. *There exist C, a such that for any open set $U \subset \mathbf{R}^3$ and $v \in H^1(U, \mathbf{R}^2)$,*

$$\begin{aligned}\int_U \Big|\phi \wedge Jv\Big| \; &\leq C \int_U |\phi| \frac{g_\varepsilon(v)}{|\log \varepsilon|} \\ &+ C\varepsilon^a (1 + \|\phi\|_{W^{1,\infty}}) \Big(\varepsilon^a + \|\phi\|_\infty + \int_{\mathrm{supp}\,\phi} (1 + |\phi|) g_\varepsilon(v)\Big) \end{aligned} \tag{6.45}$$

for all functions $\phi \in C_c^{0,1}(U, \Lambda^1\mathbf{R}^3)$.

This result is proved in [85] and is a refinement of earlier estimates of [86]. It expresses how the Jacobian is controlled by the energy g_ε.

Another lemma, stated for this problem in [85], and whose results were first proved in [86] for the case $m_\varepsilon = 1$ and in more generality in [137], is the following:

Lemma 6.10. *Assume that K is a compact subset of $\mathcal{D}$ and $v_\varepsilon \in H^1(\mathcal{D}, R^2)$ a sequence of functions such that*

$$\frac{1}{|\log \varepsilon|} \int_K g_\varepsilon(v_\varepsilon) \leq C m_\varepsilon \tag{6.46}$$

for some sequence of numbers m_ε such that $1 \leq m_\varepsilon \leq C|\log \varepsilon|$. Then $\star J v_\varepsilon / m_\varepsilon$ is precompact in the dual norm $C_c^{0,\alpha}(K)^*$ for every $\alpha > 0$. If $m_\varepsilon = 1$, then $(1/\pi)J$ is 1-rectifiable and without boundary in K. Moreover, if J is any limit of a subsequence $\star J v_\varepsilon / m_\varepsilon$, then J has finite mass in K. If in addition, μ is a nonnegative measure such that

$$\frac{g_\varepsilon(v_\varepsilon)}{m_\varepsilon |\log \varepsilon|}\, dx \to \mu \text{ weakly in measure, then } \|J\| \ll \mu, \ \frac{d\|J\|}{d\mu} \leq 1 \text{ a.e.}, \tag{6.47}$$

where $\|J\|$ denotes the total variation measure associated with J. For any 1-rectifiable current with no boundary and finite mass in $\mathcal{D}$, there exists a sequence of functions $v_\varepsilon \in H^1(\mathcal{D}, R^2)$ such that $\star J v_\varepsilon \to J$ in $\cup_{\alpha > 0} C_c^{0,\alpha}(K)^*$ and $g_\varepsilon(v_\varepsilon)/\pi|\log \varepsilon|\, dx$ converges to $\|J\|$ weakly in measure as ε tends to 0.

Construction of $\tilde{J}_\varepsilon v_\varepsilon$

Let $b \in (0, 1/6)$, $\delta = \varepsilon^b$ and

$$\mathcal{D}_\delta = \{x \in \mathcal{D} \text{ s.t. } \rho_{\mathrm{TF}}(x) > \|\nabla \rho_{\mathrm{TF}}\|_\infty \delta\}. \tag{6.48}$$

Lemma 6.8 implies that

$$1 - \frac{\rho_{\mathrm{TF}}}{f_\varepsilon^2} \to 0 \text{ in } \mathcal{D}_\delta. \tag{6.49}$$

We want to approximate $J v_\varepsilon / m_\varepsilon$ in order to be able to control it, by a mollified and boundary regularized current $\tilde{J}_\varepsilon v_\varepsilon$.

Given $\phi \in C_c^\infty(\mathcal{D}, \Lambda^1 \mathbf{R}^3)$, we claim that there exist functions $\phi_\varepsilon^1, \phi_\varepsilon^2, \phi_\varepsilon^3$ with $\phi = \phi_\varepsilon^1 + \phi_\varepsilon^2 + \phi_\varepsilon^3$, and:

1. ϕ_ε^1 is supported in $\mathcal{D}_\delta$, $\phi \mapsto \phi_\varepsilon^1$ is linear, $d\phi_\varepsilon^1 = (d\phi)_\varepsilon^1$, and

$$\left\| \frac{\phi_\varepsilon^1}{f_\varepsilon^2} \right\|_\infty + \varepsilon^\gamma \|\nabla \phi_\varepsilon^1\|_\infty \le C \left\| \frac{\phi}{\rho_{\mathrm{TF}}} \right\|_\infty. \tag{6.50}$$

2. ϕ_ε^2 and ϕ_ε^3 are error terms satisfying

$$\varepsilon^{-\gamma} \left\| \frac{\phi_\varepsilon^2}{f_\varepsilon^2} \right\|_\infty + \|\nabla \phi_\varepsilon^2\|_\infty \le C \left(\left\| \frac{\phi}{f_\varepsilon^4} \right\|_\infty + \left\| \frac{\nabla \phi}{f_\varepsilon^2} \right\|_\infty \right), \tag{6.51}$$

$$\left\| \frac{d\phi_\varepsilon^3}{f_\varepsilon^2} \right\|_{L^4} \le C \varepsilon^{\gamma/4} \left(\left\| \frac{\phi}{f_\varepsilon^4} \right\|_\infty + \left\| \frac{\nabla \phi}{f_\varepsilon^2} \right\|_\infty \right). \tag{6.52}$$

This claim is proved in [85]. The error term ϕ_ε^2 arises from mollification (convolution by a smoothing kernel) and ϕ_ε^3 from boundary regularization. Then we define the current $\tilde{J}_\varepsilon v_\varepsilon$ by

$$\tilde{J}_\varepsilon v_\varepsilon(\phi) := \star J v_\varepsilon(\phi_\varepsilon^1) \quad \forall \phi \in C_c^\infty(\mathcal{D}, \Lambda^1 \mathbf{R}^3). \tag{6.53}$$

In order to satisfy $\partial(\tilde{J}_\varepsilon v_\varepsilon) = 0$, the modification of the initial current near $\partial \mathcal{D}$ cannot be carried out by simple multiplication by a cutoff function. The boundary regularization in effect sends small vortices, which were near the boundary, away from the condensate where they do not interact with the test function ϕ.

We want to check that (6.35) is satisfied. By the properties of ϕ_ε^1 and the fact that $\partial(\star J v_\varepsilon) = 0$, we deduce

$$\partial \tilde{J}_\varepsilon v_\varepsilon(\phi) = \tilde{J}_\varepsilon v_\varepsilon(d\phi) = \star J v_\varepsilon((d\phi)_\varepsilon^1) = \star J v_\varepsilon(d\phi_\varepsilon^1) = \partial(\star J v_\varepsilon)(\phi_\varepsilon^1) = 0$$

for all compactly supported 1-forms ϕ. In order to get the rest of (6.35), we are going to prove that

$$\left| \int_{\mathcal{D}} \psi \wedge \frac{J v_\varepsilon}{m_\varepsilon} \right| \le C \left\| \frac{\psi}{f_\varepsilon^2} \right\|_\infty + C \varepsilon^\gamma (1 + \|\nabla \psi\|_\infty)(1 + \|\nabla \psi\|_\infty). \tag{6.54}$$

This implies (6.35), since for any smooth compactly supported 1-form ϕ with $\|\phi/\rho_{\mathrm{TF}}\|_\infty \leq 1$, (6.54) and (6.50) imply that

$$\Big|\frac{\tilde{J}_\varepsilon v_\varepsilon}{m_\varepsilon}(\phi)\Big| = \Big|\int_{\mathcal{D}} \phi^1_\varepsilon \wedge \frac{J v_\varepsilon}{m_\varepsilon}\Big| \leq C,$$

which implies a uniform bound on $M_{\rho_{\mathrm{TF}}}(\tilde{J}_\varepsilon v_\varepsilon/m_\varepsilon)$ by the definition (6.26).

To prove (6.54), we define $\bar{\varepsilon} = c\varepsilon^{1-(\gamma/2)}$, where c is chosen such that $\bar{\varepsilon}^2 \geq \varepsilon^2/f_\varepsilon^2$ in $\mathcal{D}_\delta$. This is possible thanks to (6.49). The choice of $\bar{\varepsilon}$ implies that

$$\int_{\mathcal{D}_\delta} f_\varepsilon^2 g_{\bar{\varepsilon}}(v_\varepsilon) \leq G_{\eta_\varepsilon}(v_\varepsilon). \tag{6.55}$$

For ψ with compact support in $\mathcal{D}_\delta$, we deduce that

$$\int |\psi| \frac{g_{\bar{\varepsilon}}(v_\varepsilon)}{m_\varepsilon |\log \bar{\varepsilon}|} \leq C \Big\|\frac{\psi}{f_\varepsilon^2}\Big\|_\infty \int_{\mathcal{D}_\delta} f_\varepsilon^2 \frac{g_{\bar{\varepsilon}}(v_\varepsilon)}{m_\varepsilon |\log \bar{\varepsilon}|} \leq C \Big\|\frac{\psi}{f_\varepsilon^2}\Big\|_\infty. \tag{6.56}$$

We use (6.45) with ε replaced by $\bar{\varepsilon}$ and (6.56) to get

$$\Big\|\int \psi \wedge \frac{J v_\varepsilon}{m_\varepsilon}\Big\| \leq C \Big\|\frac{\psi}{f_\varepsilon^2}\Big\|_\infty + C\varepsilon^{a(1-(\gamma/2))}(1 + \|\nabla\psi\|_\infty)\varepsilon^{-\gamma}|\log \varepsilon|(1 + \|\psi\|_\infty). \tag{6.57}$$

If $\gamma < a/4$, this proves (6.54).

The proof of the convergence of currents relies on

$$\Big|\frac{1}{m_\varepsilon}(\star J v_\varepsilon - \tilde{J}_\varepsilon v_\varepsilon)(\phi)\Big| = \Big|\int (\phi_\varepsilon^2 + \phi_\varepsilon^3) \wedge \frac{J v_\varepsilon}{m_\varepsilon}\Big| \leq C\Big(\Big\|\frac{\phi}{f_\varepsilon^4}\Big\|_\infty + \Big\|\frac{\nabla\phi}{f_\varepsilon^2}\Big\|_\infty\Big).$$

The last estimate comes from (6.54) and (6.51) for the part related to ϕ_ε^2. To control the integral for ϕ_ε^3, one needs an integration by parts to obtain that it is equal to $\left(\frac{1}{2}\right) | \int d\phi_\varepsilon^3 \wedge (i v_\varepsilon, \nabla v_\varepsilon)|$ and estimate it using (6.52), Holder's inequality, and (6.39).

Compactness and properties of limiting currents

The second estimate of (6.35) implies, using a measure theory lemma, compactness in some norm that we have not introduced here. This norm is stronger than $\|.\|_\varepsilon$, so that the last inequality of (6.35) implies that $\star J v_\varepsilon/m_\varepsilon$ is precompact as a sequence of distributions.

We assume that J is the limit of a convergent subsequence, still denoted by $\star J v_\varepsilon/m_\varepsilon$. It follows from (6.35), and the arguments just mentioned above, that $\tilde{J}_\varepsilon v_\varepsilon/m_\varepsilon$ converges to J in the sense of distributions, and in fact also in some local norms. The first two properties of (6.35) are inherited by J. Additional properties of J are a consequence of Lemma 6.10.

Proof of (6.36)

We assume that $\star J v_\varepsilon/|\log\varepsilon|$ converges to J and use the splitting of the energy (6.41) to get the lower bound. We have to prove that

$$\liminf_{\varepsilon\to 0} \frac{1}{m_\varepsilon|\log\varepsilon|} G_{\eta_\varepsilon}(v_\varepsilon) \geq \int_{\mathcal{D}} \rho_{\mathrm{TF}} d\|J\|, \tag{6.58}$$

$$\lim_{\varepsilon\to 0} \frac{1}{m_\varepsilon|\log\varepsilon|} I^1_{\eta_\varepsilon}(v_\varepsilon) = -\bar{\Omega} J(\rho_{\mathrm{TF}}^2 dx^3). \tag{6.59}$$

For the last term in the energy splitting, we easily get

$$|I^2_{\eta_\varepsilon}(v_\varepsilon)| \leq C\Omega_\varepsilon^2 \|f_\varepsilon^2(1-|v_\varepsilon|)^2\|_2 \leq C\varepsilon\Omega_\varepsilon^2 (G_{\eta_\varepsilon}(v_\varepsilon))^{1/2} \leq C\varepsilon|\log\varepsilon|^3. \tag{6.60}$$

Hence (6.36) follows from (6.58), (6.59), and (6.60).

The proof of (6.58) is performed on compact subsets K and thus uses a lower bound for f_ε there and the convergence of the currents.

The proof of (6.59) relies on the equation (6.32) satisfied by S_0. In order to exploit this equation, we rewrite f_ε^2 in $I^1_{\eta_\varepsilon}$ as

$$\chi_\varepsilon \rho_{\mathrm{TF}} + \chi_\varepsilon(f_\varepsilon^2 - \rho_{\mathrm{TF}}) + (1-\chi_\varepsilon) f_\varepsilon^2, \tag{6.61}$$

where χ_ε is a smooth function such that

$$\chi_\varepsilon = 1 \text{ in } \mathcal{D}_{2\delta} \text{ and } \chi_\varepsilon = 0 \text{ in } \mathcal{D}_\delta, \quad |\nabla\chi_\varepsilon| \leq \frac{C}{\delta},$$

and $\mathcal{D}_\delta$ is given by (6.48).

The last two terms arising from (6.61) are easily estimated as small errors. After using (6.32), integrating by parts, and estimating a small error term, one finds that the main contribution to $I^1_{\eta_\varepsilon}$ is

$$\frac{-1}{1+\alpha^2} \int_{\mathcal{D}} \chi_\varepsilon \rho_{\mathrm{TF}}^2 e_z \cdot (i v_\varepsilon, \nabla v_\varepsilon).$$

Since $\|\chi_\varepsilon \rho_{\mathrm{TF}}^2 dx_3\|_\varepsilon \leq C$, we can use the third estimate in (6.35) to replace $J v_\varepsilon$ by $\tilde{J}_\varepsilon v_\varepsilon$. It is then not hard to conclude the proof using the second estimate of (6.35). We refer to [85] for details.

The only point at which the proof fundamentally uses the assumption that the trapping potential is harmonic is the estimate

$$\|\chi_\varepsilon \boldsymbol{\Xi}\|_\varepsilon \leq C \text{ for } \boldsymbol{\Xi} = \frac{-\rho_{\mathrm{TF}}^2}{2(1+\alpha^2)} e_z, \tag{6.62}$$

at the end of the argument. For more general ρ_{TF} and corresponding stream functions $\boldsymbol{\Xi}$ defined by (6.10), the argument works as soon as $|\boldsymbol{\Xi}| \leq C\rho_{\mathrm{TF}}^2$ near $\partial\mathcal{D}$. Here, this is immediate from the explicit form of $\boldsymbol{\Xi}$. Other complications can arise, however, if ρ_{TF} is such that $\mathcal{D} = \{\rho_{\mathrm{TF}} > 0\}$ is not simply connected, or if $\nabla\rho_{\mathrm{TF}}$ vanishes near $\partial\mathcal{D}$.

Proof of (6.37)

We need to establish the existence of sequences for which the lower bound is achieved when $m_\varepsilon = 1$. Let us give the proof in the case that J has finite mass, the general case following by diagonalization arguments and considering the appropriate norms on currents (see [85]). If J has finite mass, one can apply Lemma 6.10, and get that there is a sequence v_ε such that $\star J v_\varepsilon \to J$ and

$$\frac{1}{|\log \varepsilon|} G_{\eta_\varepsilon}(v_\varepsilon) \leq \int_{\mathcal{D}} \rho_{\mathrm{TF}} \frac{g_\varepsilon(v_\varepsilon)}{|\log \varepsilon|} \to \int_{\mathcal{D}} \rho_{\mathrm{TF}} d\|J\|.$$

We also have (6.59) and (6.60) and thus get

$$\limsup_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|}(E(u_\varepsilon) - E(\eta_\varepsilon)) = \limsup_{\varepsilon\to 0} \frac{1}{|\log \varepsilon|} G_{\eta_\varepsilon}(v_\varepsilon) + \frac{1}{|\log \varepsilon|} I^1_{\eta_\varepsilon}(v_\varepsilon) \leq E[J].$$

The opposite inequality has been proved in the previous step.

Proof of Theorem 6.4

Since $E_\varepsilon(u_\varepsilon) \leq E_\varepsilon(\eta_\varepsilon)$, we derive from (6.41) that

$$G_{\eta_\varepsilon}(v_\varepsilon) \leq \left| I^1_{\eta_\varepsilon}(v_\varepsilon) + I^2_{\eta_\varepsilon}(v_\varepsilon) \right|. \tag{6.63}$$

We deduce from the Cauchy–Schwarz inequality that

$$|I^1_{\eta_\varepsilon}(v_\varepsilon)| \leq \int_{\mathcal{D}} \frac{1}{4} f_\varepsilon^2 |\nabla v_\varepsilon|^2 + C\Omega_\varepsilon^2 f_\varepsilon^2 |v_\varepsilon|^2 \leq \frac{1}{2} G_{\eta_\varepsilon}(v_\varepsilon) + C\Omega_\varepsilon^2(1 + \varepsilon^2\Omega_\varepsilon^2). \tag{6.64}$$

We get from (6.60), (6.63), and (6.64) that $G_{\eta_\varepsilon}(v_\varepsilon) \leq C|\log \varepsilon|^2$, which matches the hypotheses of Theorem 6.5 with $m_\varepsilon = |\log \varepsilon|$. Thus there exists a current J such that $\star J v_\varepsilon / m_\varepsilon$ converges to J in the sense of distributions, and

$$E[J] \leq \frac{1}{m_\varepsilon |\log \varepsilon|}(E_\varepsilon(u_\varepsilon) - E_\varepsilon(\eta_\varepsilon)).$$

Since $\bar{\Omega} < \bar{\Omega}_1$, this implies that $J = 0$. To finish the proof, we have to get that $m_\varepsilon = 1$. Let us assume that this is not the case, and define $m_\varepsilon = \max\{1, G_{\eta_\varepsilon}(v_\varepsilon)/|\log \varepsilon|\}$. Thus

$$0 \geq \liminf \frac{1}{m_\varepsilon |\log \varepsilon|}(G_{\eta_\varepsilon}(v_\varepsilon) + I^1_{\eta_\varepsilon}(v_\varepsilon) + I^2_{\eta_\varepsilon}(v_\varepsilon)) = 1.$$

The last inequality is a consequence of (6.59) and (6.60) since $J = 0$ and provides a contradiction.

6.4 Single Vortex line, study of $E[\gamma]$

The previous section has allowed us to obtain as a Γ limit a reduced energy for rectifiable 1-dimensional currents with locally finite mass. These currents, as explained in [85], can be identified as a countable sum of oriented Lipschitz curves γ_i, that is, $T = \sum_i T_{\gamma_i}$. We will see that the minimization of $E[T]$ can thus be reduced to a planar problem for an oriented Lipschitz curve γ.

In this section, we will be more precise about the setting of minimization. Then we will analyze the critical velocity $\bar{\Omega}_1$ and the shape of minimizers, proving that the straight vortex is not a local minimizer under certain conditions. Finally, we will study various types of critical points of $E[\gamma]$.

The analysis described in this section relies on [14, 15]. We use the notation of (6.4) and for an oriented curve γ such that $\gamma(0) = \gamma(1)$ or $\gamma(0), \gamma(1) \in \partial\mathcal{D}$,

$$E[\gamma] = \int_0^1 \rho_{\mathrm{TF}}(\gamma(s))|\dot{\gamma}|\,ds - \bar{\Omega}\int_0^1 \rho_{\mathrm{TF}}^2(\gamma(s))\dot{\gamma}\cdot e_z\,ds.$$

6.4.1 Setting of minimization of $E[\gamma]$

Theorem 6.11. *If $\alpha \geq 1$, then the energy $E[\gamma]$ is minimized when the vortex line γ lies in the yz plane, that is, the plane closest to the axis.*

Indeed, if we have a curve γ parameterized as $\gamma(t) = (x(t), y(t), z(t))$, then we can define the new curve $\tilde{\gamma}(t) = (0, \tilde{y}(t), \tilde{z}(t))$ by $\tilde{z}(t) = z(t)$ and $\tilde{y}(t) = -\sqrt{x^2/\alpha^2 + y^2}$. Then $\rho_{\mathrm{TF}}(\gamma(t)) = \rho_{\mathrm{TF}}(\tilde{\gamma}(t))$. Since $\alpha \geq 1$, $\dot{\tilde{y}}^2 \leq \dot{x}^2 + \dot{y}^2$, we have $\rho_{\mathrm{TF}}(\tilde{\gamma})|\dot{\tilde{\gamma}}| - \bar{\Omega}\rho_{\mathrm{TF}}(\tilde{\gamma})\dot{\tilde{z}} \leq \rho_{\mathrm{TF}}(\gamma)|\dot{\gamma}| - \bar{\Omega}\rho_{\mathrm{TF}}(\gamma)\dot{z}$. It follows that the energy of the new curve $E[\tilde{\gamma}]$ is less than or equal to $E[\gamma]$. If $\alpha = 1$, that is, the cross section is a disc, then our arguments imply that the vortex line is planar, but of course all transversal planes are equivalent. □

If $T = \sum_i T_{\gamma_i}$ is a minimizer, then by the previous theorem, all γ_i lie in the same plane; hence the problem reduces to planar Lipschitz curves.

From now on, we will assume that the curve lies in the plane yz, so that $x = 0$, and we denote by ρ the value of ρ_{TF} that depends only on y and z. In studying minimizers it therefore suffices to consider vortices in the domain

$$\mathcal{D}_0 := \{(y,z) : \rho(y,z) > 0\},\ \ \rho(y,z) = \rho_0 - y^2 - \beta^2 z^2 \text{ for } (y,z) \in \mathcal{D}_0. \quad (6.65)$$

The variational problem in $\mathcal{D}_0$ can be rewritten by considering only vortices of the form $\gamma = \partial U$, where $U \subset \mathcal{D}_0$ is a set of locally finite perimeter and ∂U is oriented in the standard way, so that Stokes' theorem holds. (Strictly speaking γ is the reduced boundary $\partial_* U$, see [70].) If we write χ for the characteristic function of such a set U, then

$$L[\chi] = \int_{\partial U} \rho^2 dz = \int_U 2\rho\rho_y\,dy\,dz = 2\int_{\mathcal{D}_0} \chi\rho\rho_y \text{ and } H[\chi] = \int \rho|\nabla\chi|. \quad (6.66)$$

We therefore define $\mathcal{A} = \{\chi : \mathcal{D}_0 \to \{0, 1\},\ \chi \in BV_{\text{loc}}\}$ and for $\chi \in \mathcal{A}$ we write

$$E_\Omega[\chi] = H[\chi] - \Omega L[\chi] = \int_{\mathcal{D}_0} \rho|\nabla\chi| - 2\Omega \int_{\mathcal{D}_0} \chi\rho\rho_y.$$

Then $E_\Omega[\chi] = E[\gamma]$ when $U = \{(y, z) \in \mathcal{D}_0 : \chi(y, z) = 1\}$ and $\partial U = \gamma$. In the rest of this chapter, we will write Ω instead of $\overline{\Omega}$; we restrict our attention to the yz plane and study the problem of minimizing E_Ω in $\mathcal{A}$.

Proposition 6.12. *(i) For every $\Omega \geq 0$, there exists a minimizer of E_Ω in $\mathcal{A}$. Any minimizer is either the vortex-free state $\chi = 0$, or has a vortex parallel to the z axis ($\chi = 1$ in $y < 0$) or is supported in the set $\{y < 0\}$ and is bounded away from the z axis.*

(ii) For every $\Omega \geq 0$, there exists a minimizer of E_Ω in $\{\chi \in \mathcal{A} : \chi(y, z) \neq \chi(-y, -z)$ a.e.$\}$. For any such minimizer, the associated curve γ solves the Euler–Lagrange equations for the line energy.

(iii) For every ℓ such that $\mathcal{A}_\ell := \{\chi \in \mathcal{A} : L[\chi] = \ell\}$ is nonempty, a minimum $h(\ell)$ of $H[\chi]$ in $\mathcal{A}_\ell$ is achieved.

Proof: (i) The existence of minimizers follows from standard facts about BV functions: $E_\Omega[\chi]$ is bounded below for χ in $\mathcal{A}$, and taking a minimizing sequence, we can pass to the limit and obtain convergence to a minimizer. In [85], it is shown that any such minimizer can be identified with a local minimizer in a suitable sense of $E[T]$.

Note that if $\gamma = (y(t), z(t))$ is a curve in $\mathcal{D}_0$ and if $\tilde{\gamma} = (-|y(t)|, z(t))$, then $E_\Omega[\gamma] = E_\Omega[\tilde{\gamma}]$. So we may assume that $y(t) \leq 0$ for all t for γ minimizing E. By regularity, it follows that if $y(t_0) = 0$ at some t_0, then $y'(t_0) = 0$. Then the Euler–Lagrange equations imply that $y(t) = 0$ for all t, and hence that γ is the straight vortex. If this does not hold, then regularity implies that γ is bounded away from the z-axis.

Existence of minimizers in cases (ii) and (iii) follows by exactly the same arguments, once one observes that the constraints are preserved by L^1 convergence. In case (ii), the curve γ associated with a minimizer χ must pass through the origin. It is easily seen that γ solves the Euler–Lagange equations away from the origin, and at the origin the Euler–Lagrange equations are satisfied if and only if the curvature vanishes, which must occur due to symmetry. Regularity follows from standard theory; see for example [71]. $\square$

The minimization in (ii) gives rise to curves γ that pass through the origin and are called S vortices. They are never global minimizers of E_Ω but are observed experimentally [132]. They exist whatever the value of Ω, since the vortex-free solution never satisfies the constraint. On the other hand, if Ω is small, there are no U vortices that are critical points of the energy, as we will see below.

The shape of the U vortex and its preferred location in the yz plane can be analyzed using the approximate energy $E[\gamma]$. There also exist U-vortex solutions of the Euler–Lagrange equation in the energetically less favorable xz plane, but there are no critical point of the energy in any other plane. In fact, if γ is not in the xz or yz

plane or is not planar, then one can construct small perturbations of γ that preserve ρ_{TF} and lower the energy. This implies that γ cannot be a critical point of the energy because the gradient is not zero. Of course, if the ellipticity of the cross section is small, the gradient is small, which may allow us to observe these configurations. The same reasoning holds for the upper or lower part of an S vortex, since they can be matched at the origin.

The energy of the vortex-free solution is zero. Thus, a vortex line is energetically favorable when Ω, β are such that $\inf_\gamma E[\gamma] < 0$. Recall that β determines the elongation of the trap and is included in the expression of ρ_{TF}. We will see that there exists a critical $\bar{\Omega}$ called $\bar{\Omega}_1$, with $1 < \bar{\Omega}_1 \rho_0 \leq 5/4$, such that for $\bar{\Omega} > \bar{\Omega}_1$, the vortex-free state $\chi = 0$ does not minimize E, and for β small enough, a straight line parallel to the z axis is locally unstable. We want to estimate $\bar{\Omega}_1$ and the curve γ that minimizes E. Given the constrained minimization problem $h(l)$, Ω can also be viewed as a Lagrange multiplier and $\bar{\Omega}_1$ can be defined as

$$\bar{\Omega}_1 = \inf_l \frac{h(l)}{l}. \tag{6.67}$$

6.4.2 The bent vortex

Taking the straight vortex γ_s as a test function in $E[\gamma]$ allows us to compute the critical angular velocity $\bar{\Omega}_1^s$ for which a straight vortex has a lower energy than a vortex-free solution, and we obtain $\bar{\Omega}_1^s \rho_0 = 5/4$.

For E to be negative, we need $\rho_{\mathrm{TF}} - \bar{\Omega}\rho_{\mathrm{TF}}^2$ to be negative somewhere, that is, $\bar{\Omega}\rho_0 > 1$. Hence

$$1 < \bar{\Omega}_1 \rho_0 < \frac{5}{4}.$$

We are going to look at the stability and instability of the straight vortex and prove that when the condensate has a cigar shape the first vortex is bent, while when it is a pancake, the first vortex is straight and lies on the axis of rotation.

Let us investigate the existence of a bent vortex. Notice from the expression of E that for $E[\gamma]$ to be negative, we need $\rho - \Omega\rho^2$ to be negative somewhere, that is, $\Omega\rho > 1$. For fixed Ω, we define the regions

$$\mathcal{D}_i := \{(y, z) : \Omega\rho(y, z) > 1\}, \qquad \mathcal{D}_o := \mathcal{D}_0 \setminus \mathcal{D}_i. \tag{6.68}$$

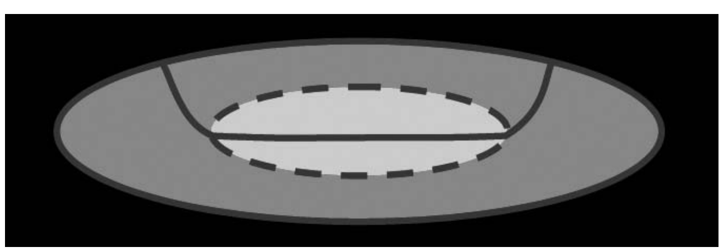


Fig. 6.3. Bent vortex.

We will refer to these sets as "the inner region" $\mathcal{D}_{\mathrm{i}}$ and "the outer region" $\mathcal{D}_{\mathrm{O}}$ respectively. In the outer region, the energy of a vortex per unit arc length is necessarily positive, since $\rho - \Omega\rho^2 > 0$, whereas in the inner region, for appropriately oriented vortices it can be negative since $\rho - \Omega\rho^2 < 0$. One can see easily that for γ to have a negative energy, part of the vortex line has to lie in the inner region, that is, close to the center of the cloud. Note that for $\mathcal{D}_i$ to be nonempty, we need at least $\Omega\rho_0 > 1$. We define $\gamma^i = \gamma \cap \mathcal{D}_i$ and $\gamma^o = \gamma \cap \mathcal{D}_o$. We assume that $\gamma(0)$ and $\gamma(1)$ lie on $\partial\mathcal{D}$.

In the region $\mathcal{D}_i$, we will see that the vortex is close to the axis for all β. On the other hand, in the region $\mathcal{D}_o$, the vortex goes to the boundary along the quickest path: if β is small, perpendicularly to the boundary, which gives rise to a bent vortex, and if $\beta > 1$, the vortex stays parallel to the axis of rotation. In [14], we prove the following:

Proposition 6.13. *For all β and all Ω, in the inner region $\mathcal{D}_i$, the straight vortex minimizes the energy restricted to $\mathcal{D}_i$, that is, $E[\gamma^i]$, for $\gamma^i = \gamma \cap \mathcal{D}_i$.*

Proposition 6.14. *For $\beta \geq 1$, in the outer region $\mathcal{D}_o$, the straight vortex minimizes the energy restricted to $\mathcal{D}_o$, that is, $E[\gamma^o]$, for $\gamma^o = \gamma \cap \mathcal{D}_o$.*

Note that in the outer region, Proposition 2 holds only for $\beta > 1$. If $\beta < 1$, the situation is somewhat more complicated: $\int_{\gamma_o} \rho\, dl$ is minimized by a path that joins $\mathcal{D}_i$ to $\partial\mathcal{D}$ along the y axis, whereas $-\int_{\gamma_o} \rho^2 dz$ is minimized by the straight vortex running along the z axis. The minimizer of the full energy reflects the competition between these two terms, and hence is bent. In particular, as a corollary of the above propositions we deduce the following:

Theorem 6.15. *For $\beta \geq 1$, $E[\gamma] \geq \inf(0, E[\gamma_s])$, where γ_s is the straight vortex parallel to the z axis. If $E[\gamma_s] < 0$, the equality can happen only if γ is the straight vortex.*

Note that for each z, there is a critical velocity $\Omega_{2d}(z)$ for the existence of a vortex in the two-dimensional section where z is constant. The region $\mathcal{D}_i$ corresponds to points z such that $\Omega > \Omega_{2d}(z)$. To prove Proposition 6.13, first note that

$$\int_{\gamma_i} \rho dl - \Omega\rho^2 dz \geq \int_{\gamma_i} \rho|dz| - \Omega\rho^2 dz \geq \int_{\gamma_i} (\rho - \Omega\rho^2) dz. \tag{6.69}$$

Since we have assumed that γ does not self-intersect, we can identify γ with the (oriented) boundary of an open set $V \subset \mathcal{D}$. Then γ_i can be identified with $\mathcal{D}_i \cap \partial V = \partial(\mathcal{D}_i \cap V) \setminus (\partial\mathcal{D}_i \cap \bar{V})$. Since $\rho - \Omega\rho^2 = 0$ precisely on $\partial\mathcal{D}_i$, this implies that

$$\int_{\gamma_i} (\rho - \Omega\rho^2) dz \; = \; \int_{\partial(\mathcal{D}_i \cap V)} (\rho - \Omega\rho^2) dz. \tag{6.70}$$

And by Stokes' theorem,

$$\int_{\partial(\mathcal{D}_i \cap V)} (\rho - \Omega\rho^2) dz \; = \; \int_{\mathcal{D}_i \cap V} (1 - 2\Omega\rho)\rho_y \, dy\, dz. \tag{6.71}$$

The definition of $\mathcal{D}_i$ implies that $1 - 2\Omega\rho < 0$, and so this integral is clearly minimized if $\mathcal{D}_i \cap V$ is just the subset of $\mathcal{D}_i$ where $\rho_y > 0$, so that

$$\int_{\partial(\mathcal{D}_i \cap V)} (\rho - \Omega\rho^2) dz \;\geq\; \int_{\{(y,z)\in\mathcal{D}_i : y<0\}} (1 - 2\Omega\rho)\rho_y \, dy \, dz. \tag{6.72}$$

Again using Stokes' theorem and the fact that $\rho - \Omega\rho^2$ vanishes on $\partial\mathcal{D}_i$, we find that this is equal to

$$\int_{-z_*}^{z_*} \Big(\rho(0,z) - \Omega\rho^2(0,z)\Big) dz, \tag{6.73}$$

where $(0, \pm z_*)$ are the points where the z axis intersects $\partial\mathcal{D}_i$. Combining these inequalities, we find that

$$\int_{\gamma_i} \rho \, dl - \Omega\rho^2 \, dz \geq \int_{-z_*}^{z_*} \Big(\rho(0,z) - \Omega\rho^2(0,z)\Big) \, dz. \tag{6.74}$$

It is easy to see that equality holds in (6.72), and hence in (6.74), exactly when γ is the straight vortex, and so we have proved Proposition 6.13.

To prove Proposition 6.14, fix γ such that γ^i is nonempty. The beginning and end of γ must lie in the outer region, and γ intersects the inner region, so γ^o must consist of at least two components. Let (a_1, b_1) denote the first such component and (a_2, b_2) denote the last, and write γ_1 and γ_2 to denote the corresponding portions of γ^o, so that γ_1 is parameterized as $\gamma_1 = (y, z)$, $(a_1, b_1) \to \mathcal{D}_o$, with $\gamma_1(a_1) \in \partial\mathcal{D}$ and $\gamma_1(b_1) \in \partial\mathcal{D}_i$. We need to show that γ_1 and γ_2 both have more energy than the corresponding parts of the straight vortex. We will consider only γ_1 since the argument for γ_2 is exactly the same.

Define $\gamma_s = (0, \zeta)$ to be a parameterization of the part of the straight vortex joining $(0, -z_{\max})$ to $(0, -z_*)$, where $z_{\max} = \sqrt{\rho_0}/\beta$:

$$\tilde{\zeta}(t) = -\frac{1}{\beta}(y(t)^2 + \beta^2 z(t)^2)^{1/2}, \;\; \zeta(t) = \max_{a \leq s \leq t} \tilde{\zeta}(s). \tag{6.75}$$

Recall that we have $\gamma_1 = (y(t), z(t))$. The definition is arranged so that $t \mapsto \zeta(t)$ is nondecreasing and $|\dot{\gamma}_s| = \dot{\zeta}$. To prove the proposition, it thus suffices to show that

$$\rho(\gamma_1)|\dot{\gamma}_1| - \Omega\rho^2(\gamma_1)\dot{z} \geq \rho(\gamma_s)|\dot{\gamma}_s| - \Omega\rho^2(\gamma_s)\dot{\zeta}. \tag{6.76}$$

If $\zeta(t) > \tilde{\zeta}(t)$, this is clear, because then $\dot{\zeta} = 0$, so the right-hand side vanishes, while the left-hand side is nonnegative, by the defining property of the outer region $\mathcal{D}_o$.

And if $\zeta(t) = \tilde{\zeta}(t)$, then $\rho(\gamma_1(t)) = \rho(\gamma_s(t))$, and so in this case $0 \leq 1 - \Omega\rho(\gamma_1(t)) = 1 - \Omega\rho(\gamma_s(t)) \leq 1$. So we need to show only that

$$|\dot{\gamma}| - c\dot{z} \;\geq\; |\dot{\gamma}_s| - c\dot{\zeta} \tag{6.77}$$

for any $c \in [0, 1]$. We will apply (6.77) to $c = \Omega\rho(\gamma_s(t))$.

To do this, first note that

$$\dot{\zeta} = \dot{\tilde{\zeta}} = \frac{1}{\tilde{\zeta}}\left(\frac{y\dot{y}}{\beta^2} + z\dot{z}\right) = (\dot{y}, \dot{z}) \cdot \left(\frac{1}{\tilde{\zeta}}(\frac{y}{\beta^2}, z)\right).$$

So

$$|\dot{\zeta}| \ \leq \ |\dot{\gamma}|\left(\frac{1}{\tilde{\zeta}^2}(\frac{y^2}{\beta^4} + z^2)\right)^{1/2} = |\dot{\gamma}|\left(\frac{\beta^{-4}y^2 + z^2}{\beta^{-2}y^2 + z^2}\right)^{1/2}.$$

Since $\beta > 1$, we conclude that $|\dot{\zeta}| \leq |\dot{\gamma}_1|$. Also, it is clear that $|\dot{z}| \leq |\dot{\gamma}_1|$. So if $0 \leq \alpha \leq 1$, then

$$|\dot{\gamma}_1| - c\dot{z} \geq |\dot{\gamma}_1|(1 - c) \geq \dot{\zeta}(1 - c) = |\dot{\gamma}_s| - c\dot{\zeta},$$

which proves (6.77), and hence Proposition 6.14. We now investigate further on the stability of the straight vortex. We parameterize the straight vortex as $\gamma_s(z) = (0, z)$ for $-z_{\max} < z < z_{\max}$, with $z_{\max} = \sqrt{\rho_0}/\beta$.

We consider perturbations of the straight vortex of the form $\gamma_\delta(z) = (\delta v(z), z + \delta^2 w(z)) + O(\delta^3)$ for $|z| < z_{\max}$. We require that w be chosen such that $\rho(\gamma_\delta(\pm z_{\max})) = 0$, thereby respecting the condition that the vortex line terminate at the boundary of the cloud.

Writing a Taylor series expansion for E, one finds that

$$E[\gamma_\delta] = E[\gamma_s] + \frac{\delta^2}{2}(v, E''[\gamma_s]v) + O(\delta^3), \tag{6.78}$$

where

$$(v, E''[\gamma_s]v) = \int_{-z_{\max}}^{z_{\max}} 2(2\Omega\rho - 1)v^2 + \rho v'^2 dz. \tag{6.79}$$

To get this it is necessary to integrate by parts and use the fact that the straight vortex solves the Euler–Lagrange equations for E. In particular, this eliminates all terms involving w. No boundary terms arise from integration by parts because $\rho(\gamma_\delta) = 0$ at the endpoints. In the case $\Omega = 0$, this equation has been studied in [152].

We say that the straight vortex is stable if $(v, E''[\gamma_s]v) > 0$ for all v, and unstable if $(v, E''[\gamma_s]v) < 0$ for some v.

Theorem 6.16. *The straight vortex is stable if*

$$\bar{\Omega}\rho_0 > \frac{3}{4} + \frac{1}{4\beta^2}. \tag{6.80}$$

The straight vortex is unstable if $\beta < 1/\sqrt{3}$ and

$$\bar{\Omega}\rho_0 < \frac{1}{6} + \frac{1}{6\beta^2}. \tag{6.81}$$

Proof: To prove the instability of the straight vortex, we will find explicit perturbations v for which $(v, E''[\gamma_s]v) < 0$. These also indicate the shape of good test functions.

We define a perturbation v (depending on a parameter θ, which for now we regard as fixed) by

$$v(z) = \begin{cases} 0 & \text{if } z \leq \theta z_{\max}, \\ \left(\frac{z}{z_{\max}} - \theta\right)(1-\theta)^{-1} & \text{if } z \geq \theta z_{\max}. \end{cases}$$

Here v is normalized so that $v(z_{\max}) = 1$. For this choice of v, a lengthy but straightforward calculation shows that

$$(v, E''[\gamma_s]v) = \frac{2\Omega\rho_0^{3/2}}{30\beta}[(1-\theta)^2(\theta+4) - \frac{5}{\Omega\rho_0}(1-\theta) - \beta^2(1+\frac{\theta}{2})] =: \frac{2\Omega\rho_0^{3/2}}{30\beta}\Delta(\theta).$$

It follows that the straight vortex is unstable if

$$(1-\theta)^2(\theta+4) < \frac{5}{\Omega\rho_0}\left((1-\theta) - \beta^2\left(1+\frac{\theta}{2}\right)\right) \tag{6.82}$$

for some $\theta \in [0, 1)$. It is helpful to write θ as $\theta = 1 - \eta\beta^2$ for some $\eta > 0$ to be determined. Then (6.82) can be written in terms of η, as

$$\Omega\rho_0 < 5\left(\frac{1 + (\beta^2/2) - (3/2\eta)}{\eta\beta^2(5 - \eta\beta^2)}\right).$$

This is satisfied if

$$\Omega\rho_0 < \frac{1 + (\beta^2/2) - (3/2\eta)}{\eta\beta^2} = \frac{1}{2\eta} + \frac{1}{\eta\beta^2}\left(1 - \frac{3}{2\eta}\right).$$

The maximum of the right-hand side is achieved for η close to 3, so we can take $\eta = 3$ to find that (6.81) is a sufficient condition for instability. Because $\theta = 1 - \eta\beta^2 \geq 0$, this conclusion holds only if $\beta \leq 1/\sqrt{3}$. For larger values of β, one can make different choices of θ to find thresholds for instability.

To derive the sufficient condition for stability, note that for every z,

$$\frac{3\rho}{2\rho_0} - \frac{(z\rho)'}{2\rho_0} = 1.$$

Multiplying v^2 by the expression on the left and integrating by parts, we obtain

$$\int_{-z_{\max}}^{z_{\max}} v^2 dz = \int_{-z_{\max}}^{z_{\max}} \rho\left[\frac{3v^2}{2\rho_0} + \frac{z}{\rho_0}vv'\right]dz.$$

Since $|z|/\rho_0 \leq z_{\max}/\rho_0 = 1/\beta\sqrt{\rho_0}$ for $|z| < z_{\max}$, we have

$$\int_{-z_{\max}}^{z_{\max}} v^2 dz \leq \int_{-z_{\max}}^{z_{\max}} \rho \left[\frac{3}{2\rho_0} v^2 + \frac{1}{\beta\sqrt{\rho_0}} |v|\,|v'| \right] dz.$$

Now we use the inequality $ab \leq a^2/2 + b^2/2$ to deduce

$$\int_{-z_{\max}}^{z_{\max}} v^2 dz \leq \int_{-z_{\max}}^{z_{\max}} \rho \left[\left(\frac{3}{2\rho_0} + \frac{1}{2\rho_0\beta^2} \right) v^2 + \frac{1}{2}(v')^2 \right] dz.$$

In particular, if

$$\Omega\rho_0 > \frac{3}{4} + \frac{1}{4\beta^2}$$

then this implies that $(v, E''[\gamma_s]v) > 0$ for all v. □

Note that the two values are consistent in the sense that they both scale like $1/\beta^2$ when β is small. For Ω large, one expects several vortices in the condensate, but the fact that a straight vortex is stable gives an indication that for Ω large, each vortex should be nearly straight, which is consistent with the observations [1]. Recall that the stabilization of the cloud requires that the rotation be not stronger than the trapping potential, which reads in our notation $\Omega < 1/\varepsilon$.

Remark 6.17. It is interesting to see what happens in Theorem 6.16 when $\bar{\Omega}\rho_0 = 5/4$, that is, when the straight vortex has zero energy. The first inequality yields that if $\beta > 1/\sqrt{2}$, then the straight vortex is stable for all Ω such that $\Omega\rho_0 > 5/4$, that is, when $E[\gamma_s] < 0$. If $\beta > 1$, we have seen that γ_s is not just stable but in fact minimizes E. The second inequality implies that if $\beta < \sqrt{2/13} \approx 0.39$, then the straight vortex is unstable at the velocity $\bar{\Omega}\rho_0 = 5/4$ at which $E[\gamma_s] = 0$. As a result, for these values of β, the first vortex to nucleate as $\bar{\Omega}$ increases is a bent vortex. Note that it has been observed in [152] that for $\beta < 1/2$, the ground state of the system exhibits a bent vortex. Numerical results of [68] also show that bent vortices are energetically favorable when β is small.

Our results imply that under certain conditions there exists a nontrivial and non-straight minimizing vortex. This minimizer is seen in experiments and is called a U vortex.

In the case $\beta < 1$, that is, when the vortex line is bent, we will prove that the vortex has a minimum length. This is related to the fact that the vortex has to go to the center of the cloud and spend some time in the inner region.

For an open set $U \subset \mathcal{D}$ with Lipschitz boundary, we endow ∂U with an orientation in the standard way, so that Stokes' theorem holds.

We will deduce a lower bound on the vortex length from the following isoperimetric-type inequality:

Theorem 6.18. *For every* $0 < \beta \leq 1$,

$$\left| \int_{\partial U} \rho^2 dz \right| \ \leq\ (2\sqrt{\rho_0})^{1/2} \left(\int_{\partial U} \rho\, dl \right)^{3/2} \tag{6.83}$$

for every connected open subset $U \subset \mathcal{D}$.

Proof: 1. We use Stokes' theorem to calculate

$$\int_{\partial U} \rho^2 dz = 2\int_U \rho\rho_y \, dy \, dz \le 2\int_{U^-} \rho\rho_y \, dy \, dz, \tag{6.84}$$

where $U^- = \{(y,z) \in U \ : \ y < 0\}$, since $\rho\rho_y \le 0$ for $(y,z) \in \mathcal{D}$ such that $y \ge 0$.

So the coarea formula implies that

$$\begin{aligned}\int_{\partial U} \rho^2 dz \ &\le 2\int_{U^-} \rho \frac{|\rho_y|}{|\nabla\rho|} |\nabla\rho| \, dy \, dz \\ &= 2\int_{\rho_*}^{\rho^*} s \left(\int_{\{(y,z)\in U^- \ : \ \rho(y,z)=s\}} \frac{|\rho_y|}{|\nabla\rho|} \, dl \right) ds,\end{aligned}$$

where $\rho_* = \inf\{\rho(y,z) \ : \ (y,z) \in U\}$, and $\rho^* = \sup\{\rho(y,z) \ : \ (y,z) \in U\}$. Thus

$$\left| \int_{\partial U} \rho^2 dz \right| \ \le \ |\rho^* - \rho_*| \quad \sup_s \left(s \int_{\{(y,z)\in U \ : \ \rho(y,z)=s\}} \frac{\rho_y}{|\nabla\rho|} \, dl \right).$$

Thus to prove the theorem it suffices to establish the following two claims:

$$s \int_{\{(y,z)\in U \ : \ \rho(y,z)=s\}} \frac{\rho_y}{|\nabla\rho|} \, dl \ \le \ \int_{\partial U} \rho \, dl \tag{6.85}$$

for every s, and

$$|\rho^* - \rho_*| \le (2\sqrt{\rho_0})^{1/2} \left(\int_{\partial U} \rho \, dl \right)^{1/2}. \tag{6.86}$$

2. We first prove (6.85). Fix some $s \in (\rho_*, \rho^*)$ and write Γ_s to denote $\{(y,z) \in U^- \ : \ \rho(y,z) = s\}$. Also, let $\tilde{\Gamma}_s$ denote $\partial U \cap \{\rho \ge s\}$.

First assume for simplicity that Γ_s is connected, so that it consists of the short arc of the ellipse $\{\rho = s\}$ joining two points, say $p_0 = (y_0, z_0)$ and $p_1 = (y_1, z_1)$ with $z_0 < z_1$. We can represent Γ_s as the image of the mapping

$$z \mapsto (y(z), z) = \left(-\left(s - \beta^2 z^2\right)^{1/2}, z \right), \qquad z_0 < z < z_1.$$

Differentiating the identity $\rho(y(z), z) = s$ we find that $\rho_y y'(z) + \rho_z = 0$. Thus

$$\left| \frac{d}{dz}(y(z), z) \right| = \left(1 + y'(z)^2\right)^{1/2} = \left(\frac{(\rho_y^2 + \rho_z^2)}{\rho_y^2} \right)^{1/2} = \frac{|\nabla\rho|}{|\rho_y|}.$$

It follows that

$$s \int_{\{(y,z)\in U \ : \ \rho(y,z)=s\}} \frac{\rho_y}{|\nabla\rho|} dl \ = \ s\int_{z_0}^{z_1} dz.$$

On the other hand, the one-dimensional measure of $\tilde{\Gamma}_s$ is certainly greater than $|p_1 - p_0| \geq z_1 - z_0$, and $\rho \geq s$ on $\tilde{\Gamma}_s$, and so

$$\int_{\tilde{\Gamma}_s} \rho(z, y) dl \ \geq \ s\, l(\tilde{\Gamma}_s) \ \geq \ s(z_2 - z_1).$$

This proves (6.85) if Γ_s is connected. If not, one can apply the same argument on each connected component of Γ_s.

3. Next we prove (6.86). Let q_* and q^* be points in ∂U such that $\rho(q_*) = \rho_*$, $\rho(q^*) = \rho^*$. Since we have assumed that U is connected, ∂U contains a path joining q_* to q^*. In fact it contains two such paths. If we write $\mathcal{P}$ to denote the set of all Lipschitz paths in $\mathcal{D}$ joining the level set $\{\rho = \rho_*\}$ and the level set $\{\rho = \rho^*\}$, it follows that

$$\int_{\partial U} \rho \, dl \ \geq 2 \inf_{\gamma \in \mathcal{P}} \int_\gamma \rho \, dl.$$

Arguments in the proof of Proposition 6.14 show that for $\beta \leq 1$, $\inf_{\gamma \in \mathcal{P}} \int_\gamma \rho \, dl$ is attained by a path that goes in a straight line along the y axis. Thus

$$\inf_{\gamma \in \mathcal{P}} \int_\gamma \rho \, dl = \int_{y^*}^{y_*} (\rho_0 - y^2) \, dy,$$

where $y_* = \sqrt{\rho_0 - \rho_*}$, $y^* = \sqrt{\rho_0 - \rho^*}$. And since $y_*, y^* \leq \sqrt{\rho_0}$,

$$\begin{aligned} \int_{y^*}^{y_*} (\rho_0 - y^2) \, dy \ &\geq \frac{1}{2\sqrt{\rho_0}} \int_{y^*}^{y_*} (\rho_0 - y^2) 2y \, dy \\ &= \frac{1}{2\sqrt{\rho_0}} \int_{\rho_*}^{\rho^*} \rho \, d\rho \\ &= \frac{1}{4\sqrt{\rho_0}} \left((\rho^*)^2 - (\rho_*)^2 \right). \end{aligned} \tag{6.87}$$

Since $b^2 - a^2 \geq (b - a)^2$ when $0 < a < b$, we deduce that (6.86) holds. □

Remark 6.19. The exponent $3/2$ is the best possible. An inequality similar to (6.83) is valid for $\beta > 1$, but the proof needs to be modified a bit. For the straight radial vortex,

$$\int_{\partial U} \rho^2 dz = \frac{16}{15} \frac{(\rho_0)^{5/2}}{\beta} \quad \text{and} \quad \int_{\partial U} \rho \, dl = \frac{4}{3} \frac{(\rho_0)^{3/2}}{\beta},$$

and so

$$\left(\int_{\partial U} \rho^2 dz \right) \left(\int_{\partial U} \rho \, dl \right)^{-3/2} \approx 0.52 \, \beta^{1/2} (\rho_0)^{1/4}.$$

This shows that the constant $(2\sqrt{\rho_0})^{1/2}$ in (6.83) is fairly close to sharp for $\frac{1}{4} \leq \beta < 1$, say.

A short calculation starting from (6.83) shows that if $E[\gamma] < 0$ then

$$\int_\gamma \rho \, dl > \frac{1}{(2\Omega^2 \sqrt{\rho_0})}. \tag{6.88}$$

We expect that even for a configuration with multiple vortices, each vortex line will satisfy a lower bound of the type (6.88). In a configuration with several vortices γ_k, the energy derived in [16] is $\sum E[\gamma_k] + I(\gamma_k, \gamma_j)$, where

$$I(\gamma_k, \gamma_j) = \int_{\gamma_k} |\log(\mathrm{dist}(x, \gamma_j)| \, dl.$$

Adding a vortex to a stable configuration with $n-1$ vortices requires

$$E[\gamma_n] + \sum I(\gamma_n, \gamma_j) < 0.$$

Since $I > 0$, this implies in particular that $E[\gamma_n] < 0$ and hence the bound on the length.

6.4.3 Properties of critical points

As we have seen above, the minimizer is a U vortex, but it does not exist as a critical point for all values of Ω:

Proposition 6.20. *If $\Omega\rho_0 < 1/2$, there cannot exist a critical point of the energy that lies in the yz half-plane $y < 0$.*

Proof: Suppose that γ is a vortex, parameterized by $\gamma(t) = (y(t), z(t))$, where y, z are smooth functions on an interval (a, b). We are going to construct a perturbation along which the energy gradient has a sign when $\Omega\rho_0 < 1/2$. For $s > 0$ define $\gamma_s(t) = (y(t)_s, z(t))$, $y_s(t) = \max\left(y(t) + s, -(\rho_0 - \beta^2 z^2(t))^{1/2}\right)$, and let $I := \{t \in (a, b) : \rho(\gamma(t)) > 0\}$. We compute

$$\frac{d}{ds} E[\gamma_s]\Big|_{s=0} = \int_{t \in I} -2y(t) \left(\sqrt{\dot{y}^2 + \dot{z}^2} - 2\Omega\rho\dot{z} \right) dt.$$

If γ stays in $y < 0$ and $\Omega\rho_0 < 1/2$, this is always positive, and thus γ_s cannot be a critical point of the energy. □

Let us define

$$\bar{\Omega}_0 = \inf_{l, l'} \frac{h(l) - h(l')}{l - l'}. \tag{6.89}$$

Then Theorem 6.18 implies that $\bar{\Omega}_0 < \bar{\Omega}_1$. It is proved in [85] that for $\Omega > \bar{\Omega}_0$, E_Ω has a local minimizer that is not straight only if β is small enough.

The following theorem explains a fact seen both in experiments and in numerical simulations, where it is observed that as Ω increases, the area in the yz plane enclosed by the vortex curve increases, so that the curves get closer to the z axis.

Theorem 6.21. *Fix $\Omega_1 < \Omega_2$, and for $i = 1, 2$, let χ_i minimize E_{Ω_i} in $\mathcal{A}$, and let U_i be the support of χ_i. Then $U_1 \subset U_2$.*

Theorem 6.22. *For almost every $\Omega \geq 0$, there is a unique minimizer of E_Ω in $\mathcal{A}$.*

Our results require some preliminary definitions and lemmas.

Let $\bar{\ell} := \max\{L[\chi] : \chi \in \mathcal{A}\}$. Let $\partial h(\ell)$ denote the subgradient of h at ℓ, where h is defined in Proposition 6.12,

$$\partial h(\ell) = \{\Omega > 0 : h(\ell') \geq h(\ell) + \Omega(\ell' - \ell) \quad \text{for all } 0 \leq \ell' \leq \bar{\ell}\},$$

and let h_c denote the convex envelope of h on the interval $[0, \bar{\ell}]$, that is,

$$h_c(\ell) \ = \sup\{u(\ell) : \ u \leq h, \ u \text{ is convex on } [0, \bar{\ell}]\}.$$

Finally, define $\Sigma \ = \ \{\ell \in [0, \bar{\ell}] \ : \ \partial h(\ell) \text{ is nonempty}\} \ = \ \{\ell \in [0, \bar{\ell}] : h(\ell) = h_c(\ell))\}$. Note that $\Omega \in \partial h(\ell)$ if and only if $\ell \in \Sigma$ and $\Omega \in \partial h_c(\ell)$. Theorem 6.16 implies that h is convex for ℓ close to $\bar{\ell}$ if β is sufficiently small, which ensures that Σ is nonempty. On the other hand, Theorem 6.18 shows that $h(\ell) \geq c\ell^{2/3}$. Thus we expect that h is concave near $\ell = 0$.

Lemma 6.23. *If $\chi \in \mathcal{A}$ minimizes $E_\Omega = H - \Omega L$, then $H[\chi] = h(L[\chi])$. Also, $L[\chi] \in \Sigma$, and $\Omega \in \partial h(L[\chi])$. Conversely, for any $\ell \in \Sigma$ and $\Omega \in \partial h(\ell)$, if $\chi \in \mathcal{A}_\ell$ satisfies $H[\chi] = h(\ell)$, then χ minimizes E_Ω in $\mathcal{A}$.*

To prove the first assertions, fix χ minimizing E_Ω, and let $\ell = L[\chi]$. For any $\tilde{\chi}$ such that $L(\tilde{\chi}) = \ell$, $H[\tilde{\chi}] = E_\Omega[\tilde{\chi}] + \Omega L[\tilde{\chi}] \geq E_\Omega[\chi] + \Omega L[\chi] = H[\chi]$, which proves that χ minimizes H in $\mathcal{A}_\ell$, i.e., that $H[\chi] = h(\ell)$.

To prove that $\ell \in \Sigma$, fix any $\ell' \neq \ell$ and find χ' such that $L(\chi') = \ell'$, $H(\chi') = h(\ell')$. Then $h(\ell') - \Omega\ell' = H[\chi'] - \Omega L[\chi'] = E_\Omega[\chi'] \geq E_\Omega[\chi] = h(\ell) - \Omega\ell$. Rearranging this gives $h(\ell') \geq h(\ell) + \Omega(\ell' - \ell)$, and so $\ell \in \Sigma$ and $\Omega \in \partial h(\ell)$ as claimed.

To prove the other assertions, fix $\ell \in \Sigma$ and $\chi \in \mathcal{A}$ such that $L[\chi] = \ell$, $H[\chi] = h(\ell)$, and fix $\Omega \in \partial h(\ell)$. For any $\chi' \in \mathcal{A}$, let $\ell' = L[\chi']$. Then $E_\Omega[\chi'] = H[\chi'] - \Omega\ell' \geq h(\ell') - \Omega\ell' \geq h(\ell) - \Omega\ell = E_\Omega[\chi]$. □

Lemma 6.24. *If $\chi_1, \chi_2 \in \mathcal{A}$, then for $\chi_* = \chi_1\chi_2$ and $\chi^* = \chi_1 + \chi_2 - \chi_1\chi_2$,*

$$L[\chi_1] + L[\chi_2] = L[\chi_*] + L[\chi^*], \qquad H[\chi_1] + H[\chi_2] \geq H[\chi_*] + H[\chi^*].$$

Note that if χ_i is the characteristic functions of U_i for $i = 1, 2$, then χ_* and χ^* are the characteristic functions of $U_1 \cap U_2$ and $U_1 \cup U_2$ respectively.

The first conclusion is obvious. The second follows from noting that

$$|\nabla\chi_1| + |\nabla\chi_2| \geq |\nabla(\chi_1 + \chi_2)| = |\nabla(\chi_* + \chi_*)| \ = \ |\nabla\chi_*| + |\nabla\chi^*|$$

as measures. The last equality is a consequence of the fact that $U_* := \sup\chi_*$ is a subset of $U^* := \operatorname{supp}\chi^*$. Thus if $\partial U_* \cap \partial U^*$ is a set of positive one-dimensional measure, then their outer unit normals must be parallel (rather than antiparallel) along this set. Hence there can be no cancellation. □

Proof of Theorem 6.21: Fix $\Omega_1 < \Omega_2$ and let χ_i be a minimizer of E_{Ω_i}, $i = 1, 2$. Let $\ell_i = L[\chi_i]$ for $i = 1, 2$. Define χ_* and χ^* as in Lemma 6.24 and let $\ell_* = L[\chi_*]$, $\ell^* = L[\chi^*]$. From Proposition 6.12 we know that minimizers are contained in the region where the integrand in L is positive, and it follows that $\ell_* \leq \ell_1, \ell_2 \leq \ell^*$. Moreover, to prove the theorem it suffices to show that $\ell^* = \ell_2$, since this will prove that $\chi_2 = \chi^*$, or in other words, that $U_1 \cup U_2 = U_2$, for $U_i = \operatorname{supp} \chi_i$.

To do this, write $h_* := H[\chi_*]$ and note that $h_* \geq h(\ell_*) \geq h_1 + \Omega_1(\ell_* - \ell_1)$. Similarly, $h^* = H(\chi^*) \geq h_2 + \Omega_2(\ell^* - \ell_2)$. Lemma 6.24 implies that $h_* + h^* \leq h_1 + h_2$ and that $\Omega_1(\ell^* - \ell_1) = \Omega_1(\ell_2 - \ell^*)$, and so by adding the two equations and rearranging, we find that $0 \geq (\Omega_2 - \Omega_1)(\ell^* - \ell_2)$. Since $\Omega_2 > \Omega_1$ and $\ell^* \geq \ell_2$, we deduce that $\ell_* = \ell_2$ as required. □

Proof of Theorem 6.22: First we claim that the set $M := \{\Omega > 0 : \Omega \in \partial h(\ell^*) \cap \partial h(\ell_*)$ for some $\ell_* < \ell^*\}$ is at most countable. Indeed, if $\Omega \in M$, then also $\Omega \in \partial h_c(\ell_*) \cap \partial h_c(\ell^*)$. And because h_c is convex, it follows that, in the interval $\ell_* < \ell < \ell^*$, h_c is affine with slope Ω. Clearly, there can be at most countably many values Ω with this property, proving the claim.

Now suppose that Ω is a value such that there are distinct minimizers $\chi_1 \neq \chi_2$ of E_Ω. Define χ_* and χ^* as in Lemma 6.24. In view of Lemma 6.24, $E_\Omega[\chi_*] + E_\Omega[\chi^*] \leq E_\Omega[\chi_1] + E_\Omega[\chi_2]$, and so it follows that χ_*, χ^* are also minimizers. Because (by Proposition 6.12) χ_1 and χ_2 are supported in the set $\{y < 0\}$, the form of ρ implies that $\ell_* := L(\chi_*) < L(\chi^*) =: \ell^*$. Then Lemma 6.23 implies that Ω belongs to the countable set M defined above. This proves uniqueness of minimizers away from a set of measure zero. □

For small l, we believe that there are minimizers of the constrained problem $h(l)$ that are not minimizers of $E[\chi]$ and thus provide the existence of nonminimizing, critical points of E_Ω with U shape. For β small, given the isoperimetric inequality from Theorem 6.18, which implies that $h(\ell) \geq c\ell^{2/3}$ near $l = 0$, the curve $h(l)$ should be concave in this region. On the other hand, if χ minimizes E_Ω, then $h(l)$ is locally convex near $L(\chi)$. Thus the simplest possible behaviour that we expect for the curve $h(l)$ when β is small is to be concave close to $l = 0$ and then convex.

6.5 A few open questions

6.5.1 Small velocity

The result of Theorem 6.4 is only asymptotic and one would be interested in a finite ε statement:

Open Problem 6.1 *If $\bar{\Omega} < \bar{\Omega}_1$, prove that u_ε is vortex-free for small ε.*

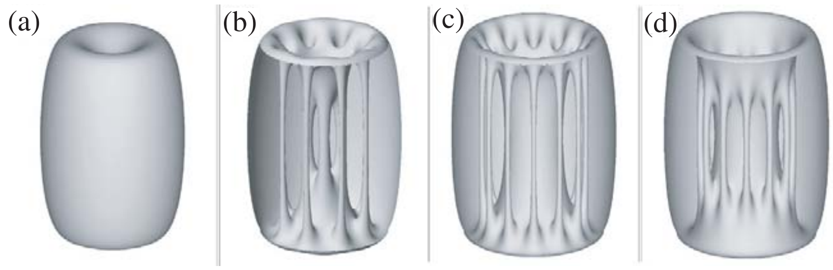


Fig. 6.4. $a = -0.1$, $b = 1.1$, $\beta = 1/7$. Side view of the condensate for $\Omega = 0.12$ (a), 0.2 (b), 0.28 (c), 0.32 (d). Isosurface of lowest density.

6.5.2 Critical points of $E_\Omega[\chi]$

Open Problem 6.2 *For fixed Ω, estimate the number and types of critical points of $E_\Omega[\chi]$. In particular, study the problem $h(l)$ for small l.*

6.5.3 Finite number of vortices

An analysis as in the two-dimensional case where the interaction energy between vortices is rigorously derived is not available at the moment for this three-dimensional problem.

6.5.4 Other trapping potentials

Our arguments depend very little on the specific geometry of $\mathcal{D}_0$ and ρ, and with small modifications would apply quite generally to families of isoperimetric-type problems. On the other hand, the derivation and shape of the line energy strongly rely on the special function ρ.

A natural open question is to consider the case of trapping potentials of the type $ar^2+br^4/16+\beta^2z^2$. Numerical simulations in Figures 6.4, 6.5 illustrate the 3D shape of vortices in this case. In Figure 6.5, in the xy plane, the condensate is an annulus with vortices located on two concentric circles. The convexity of the bending differs according to the two circles. We refer to [11] for more details.

6.5.5 Whole space problem

Open Problem 6.3 *Prove the equivalent Γ convergence result of Theorem 6.2 when the energy E_ε defined in (6.1) is posed in $\mathbf{R}^3$ instead of $\mathcal{D}$, and with the constraint that $\|u_\varepsilon\|_2 = 1$. A natural question is the shape of vortex lines: are they closed curves or do they go to infinity in the region of low density?*

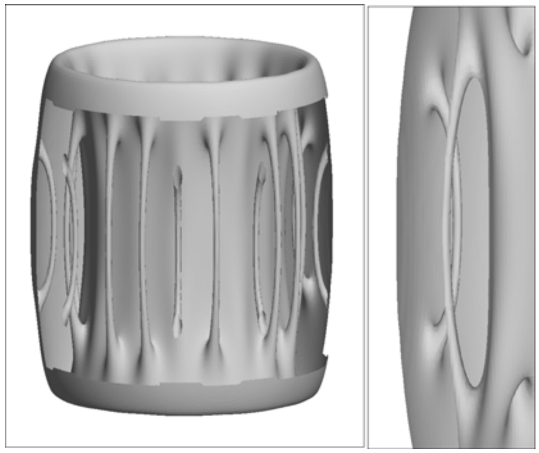


Fig. 6.5. Vortex details for $\Omega = 0.48$.

6.5.6 Decay of vortices

Numerical simulations of the time-dependent problem with Schrödinger dynamics [102] show that the long-time behaviour of the solution is that of the corresponding minimizer of the stationary problem in $\mathcal{D}$. Though the Schrödinger dynamics preserve the total energy, it turns out that the excess energy is eventually located in waves in the low-density region.

Open Problem 6.4 *Consider an initial solution of the time-dependent problem with Schrödinger dynamics, which is either a U or S vortex, and analyze its decay when the rotation Ω is equal to* 0 *or stopped slowly.*

The decay of the U vortex is expected to be similar to what is displayed in Figure 6.1. The decay of the S vortex is more mysterious, since a horizontal S still carries energy.

7

Superfluid Flow Around an Obstacle

In this chapter, we address another issue related to superfluidity: the existence of a dissipationless flow induced by the motion of a macroscopic object in a superfluid. The nucleation of vortices corresponds to the breakdown of this dissipationless phenomenon.

A classical experiment on superfluid helium consists in flowing helium around an obstacle. If the velocity c of the flow at infinity is sufficiently small, the flow is stationary and dissipationless, as opposed to what happens in a normal fluid. On the other hand, beyond a critical velocity, the flow becomes time-dependent and vortices are emitted periodically from the north and south poles of the obstacle. Numerical simulations illustrating this behaviour have been performed by Frisch, Pomeau, Rica [67] and are displayed in Figure 7.1: a pair of vortices has been emitted and is flowing behind the obstacle, while the next pair is being formed on the boundary of the obstacle. In [67], the authors have also computed the critical velocity for nucleation of vortices. Other related works, which we will describe below, include [78, 83]. The absence of dissipation at low velocity can be explained by the existence of a stationary solution to some two-dimensional nonlinear Schrödinger equation, on which we will focus. The superfluid velocity is given at any point in the flow by the gradient

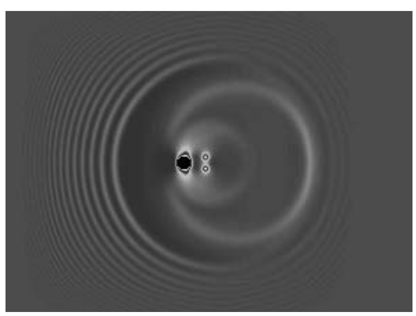


Fig. 7.1. Numerical simulation of a superfluid flow around an obstacle: the velocity at infinity is along the x axis and vortices are emitted. Courtesy of Y. Pomeau and S. Rica.

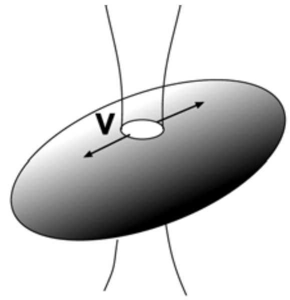


Fig. 7.2. Stirring a laser in a condensate

of the phase of the wave function: if the wave function does not vanish, then the velocity is well defined everywhere. The vortices are points where the wave function vanishes and around which the circulation of the velocity is quantized.

Very recently, an experiment was conducted at MIT by Raman et al. [128] (see also [117, 129]) in Bose–Einstein condensates, to study there the existence of a dissipationless flow. Instead of a macroscopic object, the obstacle is a blue detuned laser beam. The condensate is fixed and the obstacle is stirred in the condensate, as illustrated in Figure 7.2. Similar features to those of helium are observed, namely the evidence of a critical velocity for the onset of dissipation. The energy release is measured as a function of the velocity of the stirrer: if the velocity is small, the flow is almost dissipationless and the drag on the obstacle is very small, while above a critical value of the velocity, the flow becomes dissipative. Numerical simulations have been performed by [13, 83], relating the increase in energy dissipation to vortex nucleation. The mathematical description of the experiments is quite involved, since one has to take into account the three-dimensional geometry of the condensate and the effect of the inhomogeneous potential trapping the atoms. Thus we will study a model case, which still allows us to understand the main features of the experiments and the geometry of the problem.

In the first section, we describe the mathematical framework. The next two sections will be devoted to the proofs of two specific results.

7.1 Mathematical setting

The first part will be devoted to a two-dimensional problem modelling helium, its mathematical properties and open questions. In the following part, we will focus on a simplified problem related to the experiment on BEC, where the inhomogeneity in the density is at the origin of different behaviours.

7.1.1 Two-dimensional flow

The problem of a superfluid helium flow around an obstacle can be formulated as follows: understand the properties of the solutions ψ of

$$2i\partial_t\psi + \Delta\psi - 2ic\partial_x\psi + (\rho_0 - |\psi|^2)\psi = 0, \tag{7.1}$$

for $\mathbf{x} = (x, y)$ in $\omega = \mathbf{R}^2 \setminus \overline{B_1}$, where B_1 is a ball modelling the obstacle, and $\psi = 0$ on ∂B_1. Here c is the velocity of the flow at infinity (oriented along the x axis) and ρ_0 some fixed number. This equation is invariant under Galilean boost. Here, we choose to work in the frame where the obstacle is fixed. We are going to prove that the modulus of the solution tends to ρ_0 as $|\mathbf{x}|$ goes to infinity, but we do not put this as a hypothesis. If the flow is dissipationless, that is, for small c, we expect the existence of a stationary stable solution of this equation, while if c is increased, the flow becomes times dependent and vortices are nucleated.

Behaviour at small c

Our main result consists in a rigorous proof of the existence of stationary solutions of (7.1) for small c, such that $|\psi|$ does not vanish in ω, which implies that ψ does not have vortices. This is based on [4] and will be proved in the next section:

Theorem 7.1. *There exists $c_0 > 0$ such that for all $c \in (0, c_0)$, the problem*

$$\Delta\psi - 2ic\partial_x\psi + (1 - |\psi|^2)\psi = 0 \quad in \quad \omega = \mathbf{R}^2 \setminus \overline{B_1}, \tag{7.2}$$

$$\psi = 0 \quad on\ \{r = 1\}, \tag{7.3}$$

has a vortex-free solution ψ_c, that is, $|\psi_c| > 0$ in ω.

Let us first explain the main steps and difficulties that arise in the proof. A natural setting to prove the existence of solutions is to minimize the energy corresponding to equation (7.2), namely

$$E_c(\psi, \omega) = E_0(\psi, \omega) - c\tilde{L}(\psi, \omega), \tag{7.4}$$

where

$$E_0(\psi, \omega) = \int_\omega \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(1 - |\psi|^2)^2, \tag{7.5}$$

$$\tilde{L}(\psi, \omega) = \int_\omega (i\psi, \partial_x\psi). \tag{7.6}$$

But it turns out that for $\psi \in H^1_{\text{loc}}(\omega)$ such that $E_0(\psi) < +\infty$, the momentum term $\tilde{L}$ is not well defined, and we believe that for the solution that we will construct below, this term is not finite. Hence, we want to minimize E_c in bounded domains $\omega_R = \omega \cap B_R$ and pass to the limit as R is large. As such, it is very difficult to find good bounds on the solutions at finite R and pass to the limit. Thus we will need to do a constrained minimization to get extra information on the solutions, and then check that the constraint is not active. This will require a careful estimate of $\tilde{L}$ in terms of E_0, which is inspired by [71], where the existence of a solution of (7.2) in the whole space $\mathbf{R}^2$ for small c is derived, but no analysis of the absence of vortices is made.

A first step is to study the solution ψ_0 at $c = 0$, which is real-valued, radial, and increasing. This solution is also unique in the class of functions with finite energy E_0. The uniqueness property is crucial for our existence result for c small. Indeed, there are many solutions of (7.2)–(7.3) with $c = 0$ of the type $f(r)\exp(id\theta)$, for any integer d, but if $d \neq 0$, then E_0 is not finite. The uniqueness is obtained by a Pohozaev identity, using an idea of Mironescu [112]: the idea consists in taking the quotient of two solutions and to use the framework of the Pohozaev identity to derive that the quotient of two solutions is in fact identically equal to 1, and thus provides uniqueness.

The constraint that we prescribe on a possible solution ψ_c is that $E_0(\psi_c) - E_0(\psi_0)$ be small. The a priori estimates rely on the fact that for c small, we expect ψ_c/ψ_0 to be close to 1. This allows us to pass to the limit in R and check that the constraint is not active. This existence proof is not very far from an implicit function theorem, though we have not found the right functional space in which to apply it.

We do not prove the stability of the solution that we construct, but only that it is obtained as the limit of stable solutions (local minimizers of E_c) as R tends to infinity.

Open Problem 7.1 *For small c, the solution constructed in Theorem 7.1 is stable.*

Behaviour at large c

A natural mathematical question is to study what happens when c is large. In [78], a numerical study of the number of solutions of (7.2)–(7.3) is made as a function of c: at low velocity, there are three stationary solutions, the one minimizing the energy, which is vortex-free, but also a one-vortex and a two-vortex solution. The three branches meet at the critical value of c. For c larger than this critical velocity, there are no solutions. The rigorous mathematical description of the branches is still open. Beyond the critical velocity, the solutions of the time-dependent problem cannot be stationary (or close to a stationary solution) and vortices are emitted periodically from the obstacle, as illustrated in the numerical simulations of [67] and [78].

Open Problem 7.2 *For c large, there are no solutions of (7.2)–(7.3).*

This has been proved by Gravejat [73] when ω is replaced by $\mathbf{R}^2$, if one restricts to finite energy solutions. The proof relies on a Fourier transform, and thus, it is important to be in the whole space. His methods cannot be applied as such to our case with the obstacle.

One could hope to prove that at least there are no stable solutions for c large. Since vortices appear near the top and bottom of the obstacle, this gives an intuition on where the instability is likely to take place. One could hope to argue by contradiction, assuming that there is a branch of stable solutions for all c and constructing a specific direction (with vortices on the north and south poles) that could provide a negative Hessian. This issue has been addressed formally in [149] and the computations there and in [52] could give hints towards a rigorous proof.

For c large, we expect the time-dependent problem to give rise to solutions that emit vortices from the north and south poles of the obstacle almost periodically, with a period that decreases as $c - c_{\text{crit}}$ is increased.

Open Problem 7.3 *For large c, there exist solutions of (7.1) that are periodic in time.*

Other mathematical results related to this problem are concerned with travelling wave solutions in $\mathbf{R}^2$ [36] or $\mathbf{R}^3$ [34]. In the whole space, the situation is very different: there are vortex solutions, even at small speed. Here, at small speed, the presence of the obstacle prevents the existence of stationary stable vortices.

Another issue is the decay of solutions at infinity. The solutions that we have constructed in Theorem 7.1 have finite E_0, and in fact we will prove that they tend to 1. Gravejat [72, 74, 75] has obtained a more precise expansion of solutions in the case of $\mathbf{R}^2$.

Hydrodynamic formulation

As mentioned above, this problem was first addressed by Frisch, Pomeau, and Rica [67]. They have studied the case when the obstacle is a small disk of radius ε in the frame where the obstacle is fixed. Thus, they make the change of variable $\tilde{t} = \varepsilon t$, $\tilde{\mathbf{x}} = \varepsilon \mathbf{x}$ and the transformation $\psi = \sqrt{\rho} e^{i(\phi + c\tilde{x})/\varepsilon}$, which is now justified at small velocity since we know that ψ does not vanish by Theorem 7.1. The equation can be rewritten using the hydrodynamic formulation, which allows us to identify $\nabla\phi$ with a velocity:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho \nabla \phi) = 0, \\ \dfrac{\partial \phi}{\partial t} = \varepsilon^2 \dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \rho + \rho_0 + c^2 - |\nabla\phi|^2. \end{cases} \tag{7.7}$$

They look for stationary solutions and assume that the quantum pressure term $\varepsilon^2 \Delta\rho/\sqrt{\rho}$ is negligible, which is a kind of long-wave approximation, and are led to the following problem:

$$\operatorname{div}(\rho\nabla\phi) = 0, \quad \rho = \rho_0 + c^2 - |\nabla\phi|^2, \tag{7.8}$$

with boundary conditions $\partial\phi/\partial n = 0$ on ∂B_1, $\rho \to \rho_0$, and $\nabla\phi \to c$ at infinity. Note that the second equation in (7.8) is a Bernoulli law for this problem. The system (7.8) has the same mathematical formulation as that of a stationary irrotational flow of a compressible fluid about an obstacle. The existence of solutions for such a related subsonic problem (c small) and the nonexistence for c large were proved in [56, 146] using a fixed-point theorem.

Open Problem 7.4 *Prove rigorously the semiclassical limit: as ε tends to* 0*, if c is sufficiently small, the stationary solutions of (7.7) converge to solutions of (7.8).*

Note that the system (7.8) is elliptic as long as

$$\max |\nabla\phi|^2 < \frac{1}{3}(\rho_0 + c^2). \tag{7.9}$$

The critical velocity for vortex nucleation corresponds to the value of c such that the system turns from elliptic to hyperbolic. In [67], in analogy with the case when the operator is the Laplacian, it is assumed that the maximum of $|\nabla\phi|$ is reached on the north and south poles of the obstacle and the value is $2c$. Then, this and (7.9) yield $c^2_{\text{crit}} = \rho_0/11$, which is consistent with the numerical results. It is interesting to notice that $|\nabla\phi|$ tends to c at infinity, but its maximum value is reached on the boundary of the obstacle and is bigger than c. This is proved in [56], Chapter IV, Theorem 8.

The nucleation of vortices has been addressed in [89] using an Euler–Tricomi equation. The supersonic problems are much more involved. The rigorous study of the vortex nucleation near c_{crit} seems a challenging issue.

Lin and Zhang [101] address the difficult issue of whether time dependent solutions of (7.1) (or equivalently (7.7)) are close for small ε, to the solutions, with the same initial data, of (7.7) with $\varepsilon = 0$ (also called the compressible Euler equation). They prove it whenever the latter has a classical solution in some time interval $[0, T]$. The size of such time interval will depend on the initial data and the value of c. If the limiting equation (7.8) has a static solution (which is the case for small c as explained above [56, 146]), then there is global existence in time for (7.7) with $\varepsilon = 0$. An issue is to prove that this solution is vortex free for large time. For large c, the effect of the quantum pressure term in the nucleation of vortices is probably essential, and whatever the initial data, the solution of (7.1) should nucleate vortices.

7.1.2 Three-dimensional flow around a condensate

Experimental setup

In the MIT experiment [128, 117, 129], the condensate is cigar-shaped with the long axis along the x direction. In nondimensionalized units, the radii of the condensate in Figure 7.2 are $R_y = R_z = 0.65$ and $R_x = 2.18$. The stirring laser beam is modelled by an obstacle that is a cylinder $\mathcal{C}$ of axis z and radius $l = 0.19$. It moves along the x axis in the plane $y = 0$, as illustrated in Figure 7.2. In the actual experiment, the stirring laser is moved backward and forward. For simplicity, we will work in the frame in which the laser is stationary and assume that the condensate is flowing around the laser, ignoring the rapid turnaround. In order to model the experiment, one has to take into account the potential trapping the atoms, usually a harmonic one, such as $V(x, y, z) = \lambda^2 x^2 + y^2 + z^2$. It implies that the number ρ_0 in (7.1) has to be replaced by an inhomogeneous term: $\rho_{\text{TF}}(x, y, z) = \rho_0 - V(x, y, z)$, with $\rho_0 = 0.42$ in the experiment considered. The domain where $\rho_{\text{TF}} > 0$ is roughly the location of the condensate, since when there is no obstacle, $|\psi|^2 \approx \rho_{\text{TF}}$. Outside the obstacle, the analogue of equation (7.1) is satisfied, namely,

$$2i\partial_t\psi + \Delta\psi - 2i\tilde{v}\partial_x\psi + \frac{1}{\varepsilon^2}(\rho_{\mathrm{TF}} - |\psi|^2)\psi = 0,$$

where $\tilde{v}$ is the velocity of the stirrer and ε is a small parameter of order 10^{-3}. More precisely, $\varepsilon = (d/(8\pi Na))^{2/5}$, where d is the characteristic length of the harmonic potential, a the scattering length, and N the number of atoms in the condensate. We refer to [13] for more details.

There are two interesting regions of space: one is close to the center of the condensate $x = y = z = 0$, where $\rho_{\mathrm{TF}}(x, y, z)$ is bounded from below, and in any section where z is constant, the problem is similar to the 2D problem treated in the previous section; another interesting region is where the laser beam passes through the boundary of the condensate. This latter region can be analyzed by blowing up the boundary layer close to the obstacle, so that ρ_{TF} depends only on z [13]. The allowed domain is approximated as unbounded in the xy plane. In order to have two terms of the same order in the equation (the kinetic term $\Delta\psi$ and the potential one $(\rho_{\mathrm{TF}} - |\psi|^2)\psi$), this boundary layer must have a thickness of order $\varepsilon^{2/3}$, so that we rescale the domain with $\psi(\tilde{x}, \tilde{y}, \tilde{z}) = \varepsilon^{1/3}u(x, y, z)$, where $x = \tilde{x}/\varepsilon^{2/3}$, $y = \tilde{y}/\varepsilon^{2/3}$, and $z = (\sqrt{\rho_0} - \tilde{z})/\varepsilon^{2/3}$, $v = \tilde{v}\varepsilon^{2/3}$. The obstacle is now a cylinder of radius $a = l/\varepsilon^{2/3} = 5.6$. In the frame of the obstacle, the equation becomes

$$2i\partial_t u + \Delta u - 2iv\partial_x u + (2z\sqrt{\rho_0} - |u|^2)u = 0,\ x, y \in \mathbf{R}^2 \setminus \mathcal{C},\ z \in (0, L), \tag{7.10}$$

where L is the rescaled layer thickness. It involves a dependence in z in the potential term. The boundary conditions are

$$u = 0 \text{ on } \partial\mathcal{C} \cup \{z = 0\}, \quad u = u_{2\mathrm{D}} \text{ on } \{z = L\}, \tag{7.11}$$

where $u_{2\mathrm{D}}$ is the solution of (7.1) with ρ_0 replaced by $2\sqrt{\rho_0}L$. Far away from the obstacle, that is, for $|x|$ and $|y|$ large, we do not expect the solution to be almost constant as in the 2D case, but to be given by the solution of the first Painlevé equation

$$p'' + (2z\sqrt{\rho_0} - p^2)p = 0,\ p(0) = 0,\ p(L) = \sqrt{2\sqrt{\rho_0}L}. \tag{7.12}$$

The main difference with the 2D case is the dependence in z and the fact that z vanishes close to the boundary of the obstacle ($z = 0$). Our aim is to understand the structure of solutions of (7.10)–(7.11).

If one applies the computation of the critical velocity for the existence of a stationary solution of [67] to problem (7.10), one finds that the critical velocity is locally proportional to $\sqrt{z}$ and thus is zero, since z vanishes near the boundary. Hence vortices should appear close to the boundary for any small speed. However, experiments show the existence of dissipationless flows for small speed, and the numerical results in [13] indicate that for low velocity, there exists a stationary solution without vortices, while beyond some positive critical velocity, all solutions are nonstationary and vortices are nucleated close to $\{z = 0\}$.

Numerical results

In [13], we choose the size of the boundary layer L so that $\varepsilon^{2/3}L = 3\sqrt{\rho_0}/10$. This is based on the consideration that, on the one hand, $\varepsilon^{2/3}L$ should be suitably small so that $2z\sqrt{\rho_0}$ is a good approximation for $\rho_{\text{TF}} = \rho_0 - \tilde{z}^2$ in the boundary layer, and on the other hand, the critical velocity at $z = L$ is not too different from the critical velocity at the center of the cloud. Let us point out that the choice of the box size in z is rather arbitrary and one could imagine posing the problem for $z \in (-\infty, \infty)$.

When v is small, we find that the solution of (7.10)–(7.11) has surface oscillations near $z = 0$ but no vortices, even very close to the boundary, as illustrated in Figure 7.3. We will prove the existence of stationary solutions for v small. Formal

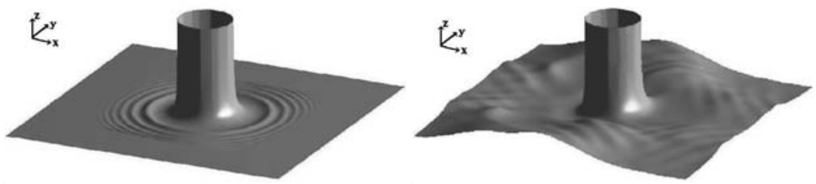


Fig. 7.3. Isosurface snapshot of $|u|$ for $v = 0.08$ and $v = 0.2$. Here $z = 0$ is the boundary of the cloud, and $z > 0$ inside the cloud.

computations allow us to understand the patterns of solutions: close to $z = 0$, it is reasonable to look for u with the ansatz

$$u(x, y, z) = p(z)\psi(x, y)e^{ivx}. \tag{7.13}$$

We can approximate $p(z)$ in this region by an Airy function given by the solution of $p'' + 2zp\sqrt{\rho_0} = 0$. Then, outside the obstacle, ψ is a solution of the 2D Helmholtz equation $\Delta\psi + v^2\psi = 0$ with $\psi = 0$ on the boundary of the disc, and $\psi \approx e^{-ivx}$ at infinity. This solution can be computed [114] in terms of Bessel functions J_k and N_k: it oscillates in space but has no vortices. These solutions are quite close to those computed close to $z = 0$. It is an open question to study the loss of stability close to $z = 0$. When v is increased, the surface oscillations develop into small handles that move up and down the obstacle without detaching, as illustrated in Figure 7.4 (the solution is periodic in time). There is no stationary solution, but no vortex shedding either: the small handles move up the obstacle to a critical z value and down. This

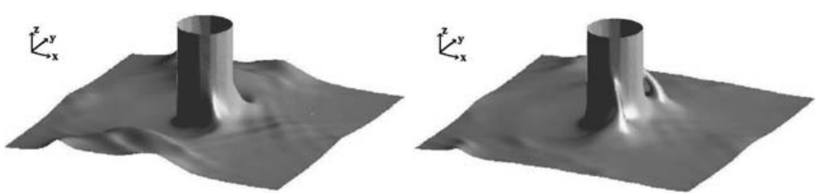


Fig. 7.4. Isosurface snapshots of $|u|$ at $t = 0.12$ and $t = 0.16$ respectively for $v = 0.24$: formation of vortex handles.

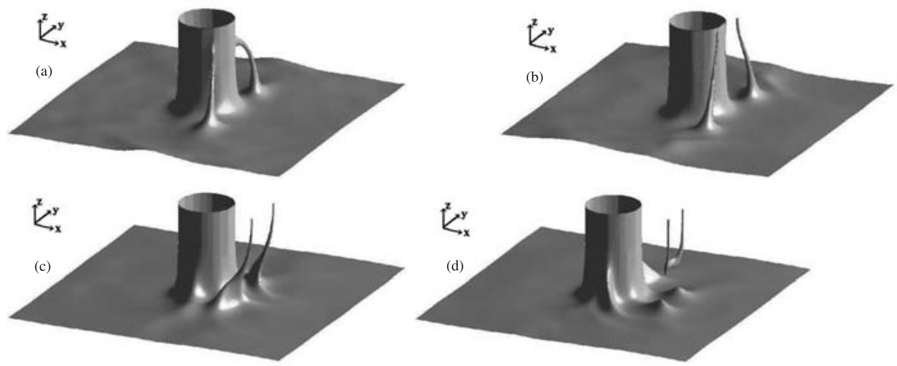


Fig. 7.5. A sequence of isosurface snapshots of $|u|$ for $v = 0.28$: (a) formation of vortex handles $t = 0.04$, (b) detachment from obstacle $t = 0.08$, (c) bending of vortex tubes $t = 0.12$ and (d) formation of vortex *half*-rings $t = 0.16$.

instability may be related to the one discussed by Anglin [24]: in our scaling, the critical velocity found in [24] is 0.2. This critical velocity corresponds to the Landau criterion.

It is only for larger velocities ($v > 0.25$) that the handles move up to the top and detach from the obstacle. This is a wholly nonlinear phenomenon, which cannot be described by a linear analysis. The vortex handles seem to first nucleate near $z = 0$ and are connected to the obstacle. As time increases, the bottom ends move away from the obstacle in a slightly downstream direction while the top end moves up along the obstacle (Figure 7.5a). When the top ends of the vortices become close to $z = L$, the bottom ends reverse their trend of moving away from obstacle. Instead, they move back to the bottom of the obstacle, as if the handles preferred certain curvatures (Figure 7.5b). Eventually, the top ends of the handle move away from the obstacle and produce a pair of vortex tubes with their bottom ends at the bottom of the obstacle (Figure 7.5c). The handles merge into a half-vortex ring, this half-ring moves both upward and downstream (Figure 7.5d). Near $z = 0$, the solution can be approximated by the solution (7.13) and this solution does not have vortices, so the instability creates the vortex but the vortex moves away. Vortex detachment happens only at sufficiently high density, in the region where the nonlinear term in the equation dominates. The direction of the vortex displacement is due to the velocity of the flow and the self-interaction of the vortex on itself, which gives a movement along its normal vector. Meanwhile, while the vortex ring starts to detach from the obstacle, another pair of vortex handles is forming near the obstacle. The above process repeats itself periodically.

Note that we have truncated the domain close to the boundary of the cloud, so that the half-ring we compute would correspond to a closed ring in the experiments.

We have to point out that the critical velocity that we have found for the onset of vortex shedding is lower than the critical velocity for the 2D problem at $z = L$. In this case, $v_{2D} = 0.35$. So the inhomogeneity in the condensate lowers the critical velocity from the 2D value. There are formal works computing the critical velocity in 2D

[113, 149], that is, taking into account the inhomogeneity in the x, y directions due to the trapping potential, but not in the z direction. One can check that for different L, the critical velocity does not change.

Rigorous results at low velocity

We want to extend the 2D result and prove the existence of stationary solutions for small v. To be consistent with the previous notation, we will set $c = v$, and for simplicity, we will set $2\sqrt{\rho_0} = L = 1$. The variables are $\mathbf{x} = (x, y, z)$, and we will set $r = \sqrt{x^2 + y^2}$ when necessary. In this setting, the stationary solutions are solutions of

$$\Delta u - 2ic\partial_x u + (z - |u|^2)u = 0 \quad \text{in} \quad \Omega = (\mathbf{R}^2 \setminus \overline{B_1}) \times (0, 1). \tag{7.14}$$

The boundary conditions are

$$u = 0 \text{ on } \{z = 0\} \text{ and } \{r = 1\}, \quad u = \psi_c \text{ on } \{z = 1\}, \tag{7.15}$$

where ψ_c is the solution of the corresponding 2D problem (7.2)–(7.3). Let us explain these boundary conditions: $\{z = 0\}$ corresponds to the outer boundary of the condensate; hence there are no atoms and the wave function vanishes. On the other side, $\{z = 1\}$ corresponds to the rescaled interior of the cloud, and the boundary condition is a stationary version of the 2D problem (7.1). The obstacle is a cylinder in the z direction of radius $r = 1$. In the next section, we will prove the following theorem based on [4].

Theorem 7.2. *There exists $c_0 > 0$ such that for all $c \in (0, c_0)$, problem (7.14)–(7.15) has a vortex-free solution u_c, that is, $|u_c| > 0$ in Ω.*

As in the proof of Theorem 7.1, we first derive properties for the solution u_0 at $c = 0$: uniqueness and nondegeneracy. An extra difficulty arises, namely, that even the first part of the energy $\mathcal{E}_0$ is not finite:

$$\mathcal{E}_0(u, \Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(z - |u|^2)^2. \tag{7.16}$$

Indeed, we will see that for r large, that is, away from the obstacle, the wave function u does not tend to some constant (as in the two-dimensional case), but behaves like $p(z)$, the solution of the following Painlevé equation:

$$p'' + (z - p^2)p = 0, \quad p(0) = 0, \quad p(1) = 1. \tag{7.17}$$

Thus, $\mathcal{E}_0(u)$ cannot be finite, since $\mathcal{E}_0(p)$ is not.

In order to overcome this difficulty, we need to introduce an energy that is finite. An idea of Mironescu [97, 112] is that all solutions for c small should have a similar behaviour at infinity. In particular, if u_0 is a solution of (7.14)–(7.15), with $c = 0$, then, if $\Omega_R = (B_R \setminus \overline{B_1}) \times (0, 1)$,

$$\lim_{R\to\infty} \mathcal{E}_0(u, \Omega_R) - \mathcal{E}_0(u_0, \Omega_R) \tag{7.18}$$

should be written in terms of a finite energy depending on u/u_0. Let us define

$$\mathcal{F}_0(w) = \int_\Omega \frac{1}{2}u_0^2|\nabla w|^2 + \frac{u_0^4}{4}(1-|w|^2)^2, \tag{7.19}$$

where $w = u/u_0$. A simple computation shows that if $\mathcal{F}_0(u/u_0)$ is finite, then the value of (7.18) is indeed $\mathcal{F}_0(u/u_0)$.

The existence part of the theorem is proved using bounded domains

$$\Omega_R = B_R \setminus \overline{B}_1 \times (0, 1),$$

and passing to the limit in R. The proof follows the same lines as in the 2D case, except that now u_0 vanishes on $z = 0$, which is a set of infinite measure on which the energy becomes degenerate. In particular, the estimate of the momentum $\mathcal{L}(w, \Omega_R) = \int_{\Omega_R} u_0^2(iw, \partial_x w)$ in terms of the energy $\mathcal{F}_0$ is more involved because near $z = 0$, the momentum density goes to zero on a set of infinite measure, and it cannot be directly estimated by the energy. This requires extra devices.

The uniqueness property of u_0 is crucial for our existence result for c small. We do not prove global uniqueness, but only in the special class of solutions with finite energy.

Open questions

Analogous open questions to the 2D case hold (Open Problems 7.1, 7.2, 7.3), in particular the nonexistence of stationary solutions for c large and the stability of the vortex-free solution. The branch of vortex-free solutions should lose stability close to $z = 0$, and a study of the linear equation in this region should help.

Open Problem 7.5 *Let c_{2D} be the critical velocity for the existence of solutions of (7.2)–(7.3). Prove that the equivalent c_{3D} defined for problem (7.2)–(7.3) is strictly smaller.*

In this section, we have not written the hydrodynamic formulation of the problem. An analogous system to (7.7) holds but it is not possible to remove the quantum pressure term, that is, the term in $\Delta\sqrt{\rho}/\sqrt{\rho}$. Indeed, this term is always dominant close to $z = 0$, which is a supersonic region. Thus it seems hard to derive information from this formulation of the equations.

7.2 Proof of Theorem 7.1

Firstly, we will prove the following properties of solutions at $c = 0$:

Theorem 7.3. *There exists a unique nontrivial nonnegative solution ψ_0 of*

$$\Delta\psi_0 + (1 - |\psi_0|^2)\psi_0 = 0 \quad in \quad \omega = \mathbf{R}^2 \setminus \overline{B}_1, \tag{7.20}$$

with boundary condition (7.3), namely

$$\psi = 0 \text{ on } \{r = 1\}. \tag{7.21}$$

It is radial increasing in r, tends to 1 as r tends to ∞; $1 - \psi_0$ and ψ_0' tend to 0 exponentially fast when r is large. If ψ is a solution of (7.20)–(7.21) in

$$X = \{\psi \in H^1_{\mathrm{loc}}(\omega, \mathbf{C}),\ E_0(\psi) < \infty\},$$

where E_0 is defined by (7.5), then ψ is equal to $\psi_0 e^{i\alpha}$, where α is a real number.

This will allow us to derive the following theorem, which will imply Theorem 7.1:

Theorem 7.4. *There exists $c_0 > 0$ such that for all $c \in (0, c_0)$, problem (7.2)–(7.3) has a vortex-free solution ψ_c, that is, $|\psi_c| > 0$ in ω. Moreover, as c tends to 0, ψ_c tends to $\psi_0 e^{i\alpha}$ in $L^\infty(\omega)$, for some α. For all M, there exists c_1 such that for $c < c_1$, up to multiplication by a complex number of modulus 1, ψ_c is the unique solution with $|E_0(\psi_c) - E_0(\psi_0)| < M$.*

Remark 7.5. Let us point out that the results still hold if instead of being the ball B_1 (respectively the cylinder $B_1 \times (0, 1)$), the obstacle is a doubly symmetric domain D (respectively $D \times (0, 1)$), star-shaped with respect to the origin, and convex in the x and y directions.

The proof relies on the fact that for c small, we expect ψ_c/ψ_0 to be close to 1, so that the energy $E_0(\psi_c) - E_0(\psi_0)$ is small. We are going to perform a constrained minimization on bounded domains, constructing approximate solutions on the sets $\omega_R = \omega \cap B_R$, and then let R go to infinity. For this purpose, we define the following energies for $w = \psi/\psi_0$:

$$F_c(w, \omega_R) = F_0(w, \omega_R) - cL(w, \omega_R), \tag{7.22}$$

where

$$F_0(w, \omega_R) = \int_{\omega_R} \frac{1}{2}\psi_0^2|\nabla w|^2 + \frac{1}{4}\psi_0^4(1 - |w|^2)^2, \tag{7.23}$$

$$L(w, \omega_R) = \int_{\omega_R} \psi_0^2(iw, \partial_x w). \tag{7.24}$$

If the domain is not mentioned, it means that the integrals are taken in the whole domain ω. We will prove the existence of $\psi_{c,R}$, a solution of the following minimization problem:

$$I_R = \inf\left\{F_c\left(\frac{\psi}{\psi_0}, \omega_R\right), \quad \psi \in H^1(\omega_R), \quad F_0\left(\frac{\psi}{\psi_0}, \omega_R\right) \le \delta\right\}, \tag{7.25}$$

where $\delta > 0$ will be made precise, and with boundary conditions

$$\psi = 0 \quad \text{on} \quad \{r = 1\}, \quad \text{and} \quad \psi = \psi_0 \quad \text{on} \quad \{r = R\}, \tag{7.26}$$

the function ψ_0 being defined in Theorem 7.3. First we show that the constraint in (7.25) is qualified, which is provided by the uniqueness result on ψ_0. Then, we show that the constraint is not active, which implies that $\psi_{c,R}$ satisfies (7.2) in ω_R. This relies on a precise estimate of the momentum L in terms of the energy F_0 (Lemma 7.11), and on the fact that if δ is chosen sufficiently small, then $F_0(\psi/\psi_0) < \delta$ implies that ψ/ψ_0 is bounded below by 1/2, and in particular does not have vortices. Many similar techniques were first developed in the context of Ginzburg–Landau problems by Bethuel, Brezis, and Helein [32, 33]. With appropriate additional bounds on $\psi_{c,R}$, we pass to the limit as R tends to infinity, to find a solution of (7.2) in ω. In order to get the convergence in $L^\infty(\omega)$, we need a precise estimate on the decrease of the energy density at infinity (Lemma 7.13), inspired by [72, 74].

7.2.1 Solutions at $c = 0$

In this subsection, we prove Theorem 7.3. We first solve (7.20) in the bounded domain $\omega_R = B_R \setminus \overline{B}_1$ to find a solution $\psi_{0,R}$ and pass to the limit as R tends to infinity: we minimize

$$E_0(\psi, \omega_R) = \int_{\omega_R} \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(1 - |\psi|^2)^2$$

among real functions, with boundary conditions $\psi = 0$ on $r = 1$ and $\psi = 1$ on $r = R$. The minimizer $\psi_{0,R}$ exists and is a positive solution of (7.20) in ω_R. The maximum principle implies that $\psi_{0,R}$ is less than 1. Moreover, by an extension of the symmetry proof of Gidas, Ni, Nirenberg [69] by W. Reichel [130], $\psi_{0,R}$ is radially increasing. Classical elliptic estimates yield uniform bounds that allow us to pass to the limit in R and obtain a positive solution ψ_0 of (7.20). At the limit, we also get that ψ_0 is radially increasing and less than 1.

We need to prove that $f = 1-\psi_0$ tends to 0 as r tends to ∞. Note that f satisfies $-\Delta f + f(1 - f)(2 - f) = 0$. Let $f_R = 1 - \psi_{0,R}$. This function satisfies the same equation as f in ω_R. There exists $k > 0$ such that for R large, $k \le \psi_{0,R}(2) \le 2k$. Hence $1 - f_R \ge k$ for $r \ge 2$, and f_R is a subsolution of

$$-\Delta f + kf = 0 \tag{7.27}$$

in $\omega_R \setminus \overline{B_2}$. Since $(1-k)\exp(-\sqrt{k}(r-2))$ is a supersolution of (7.27) in $\omega_R \setminus \overline{B_2}$, we find that for R large, $f_R(r) \le K \exp(-\sqrt{k}r)$, which is also true at the limit $R = \infty$. Returning to the equation for ψ_0, we see that elliptic estimates provide that ψ_0' goes to 0 exponentially fast at infinity. In particular, $E_0(\psi_0)$ is finite.

Now we prove that any finite energy solution ψ of (7.20)–(7.21) is equal, up to multiplication by a constant of modulus one, to the solution ψ_0.

This relies on the Pohozaev identity and ideas developped by Mironescu [112]. Let ψ be a complex-valued solution of (7.20)–(7.21) with $E_0(\psi)$ finite. The maximum principle implies that $|\psi|$ is bounded by 1 (this can be seen on the equation for $|\psi|^2 - 1$). The function $w = \psi/\psi_0$ is well defined in ω since ψ_0 does not vanish.

First we show that w is bounded: let us recall that $|\psi|$ is bounded by 1, and by Gagliardo–Nirenberg inequality (see [32] for instance), $\nabla\psi \in L^\infty(\omega)$. Since ψ_0 is a radially increasing function, one can derive from the ODE that it is concave, so

$$\forall r \in [1, 2], \quad \psi_0(r) \geq \psi_0(2)(r - 1).$$

Thus, we infer that for $r \in [1, 2]$, $|w| \leq \frac{\|\nabla\psi\|_{L^\infty}}{\psi_0(2)}$. For $r \geq 2$, we have $\psi_0(r) \geq \psi_0(2)$, so that $|w(r, \theta)|^2 \leq \frac{1}{\psi_0(2)^2}$. This proves that w is bounded.

We also have that near ∂B_1, w behaves like $(\partial\psi/\partial n)/\psi_0'(1)$: this uses Taylor expansions of ψ, $\partial_r\psi$, ψ_0, and ψ_0' near $r = 1$.

We are going to see that it is equivalent to say that $E_0(\psi)$ is finite or $F_0(w)$ is finite, where

$$F_0(w) = \int_\omega \frac{1}{2}\psi_0^2|\nabla w|^2 + \frac{\psi_0^4}{4}(1 - |w|^2)^2. \tag{7.28}$$

Indeed, let us multiply the equation for ψ_0 (7.20) by $(1 - |w|^2)$ and integrate. We find the following exact decoupling for the energy:

$$E_0(\psi, \omega_R) = E_0(\psi_0, \omega_R) + F_0(w, \omega_R) + \int_{\partial\omega_R} \frac{1}{2}\psi_0\frac{\partial\psi_0}{\partial n}(1 - |w|^2). \tag{7.29}$$

The boundary term on ∂B_1 is 0 since $|w|$ is bounded and ψ_0 is 0. The boundary term on ∂B_R tends to 0 as R tends to infinity, since $|w|$, ψ_0 are bounded and $R\psi_0'(R)$ tends to 0 exponentially. Hence, we find at the limit

$$E_0(\psi) = E_0(\psi_0) + F_0(w). \tag{7.30}$$

Thus, it is equivalent to say that $E_0(\psi)$ or $F_0(w)$ is finite. Note that this will no longer be the case in 3D.

Using the equation for ψ, we find that w is a solution of

$$\operatorname{div}(\psi_0^2\nabla w) + \psi_0^4(1 - |w|^2)w = 0 \quad \text{in } \omega. \tag{7.31}$$

Let us multiply (7.31) by $\mathbf{x}\cdot\nabla\overline{w}$, integrate in ω_R, and add the conjugate:

$$-\int_{\omega_R} 2\psi_0^2\mathbf{x}\cdot\nabla(|\nabla w|^2) + 4\psi_0^2|\nabla w|^2 + \psi_0^4\mathbf{x}\cdot\nabla\left(\frac{(1 - |w|^2)^2}{2}\right) \tag{7.32}$$

$$+\int_{\partial\omega_R} \psi_0^2\frac{\partial w}{\partial n}\mathbf{x}\cdot\nabla\overline{w} + \psi_0^2\frac{\partial\overline{w}}{\partial n}\mathbf{x}\cdot\nabla w = 0. \tag{7.33}$$

We integrate by parts the $\mathbf{x} \cdot \nabla$ terms to obtain

$$\int_{\omega_R} \psi_0^4(1-|w|^2)^2 + 2\psi_0^3(1-|w|^2)^2 \mathbf{x} \cdot \nabla \psi_0 + 4\psi_0 |\nabla w|^2 \mathbf{x} \cdot \nabla \psi_0 \tag{7.34}$$

$$= \int_{\partial \omega_R} \frac{1}{2} \psi_0^4 (1-|w|^2)^2 \mathbf{x} \cdot n \tag{7.35}$$

$$-\psi_0^2 \left(\frac{\partial w}{\partial n} \mathbf{x} \cdot \nabla \overline{w} + \frac{\partial \overline{w}}{\partial n} \mathbf{x} \cdot \nabla w \right) + \psi_0^2 \mathbf{x} \cdot n |\nabla w|^2. \tag{7.36}$$

Since the energy $F_0(w)$ is finite, we can find a sequence R_n that tends to infinity such that the boundary terms (7.35)–(7.36) on $r = R_n$ tend to 0. On $r = 1$, the boundary terms are zero, since $|w|$ is bounded, ψ_0 is zero, $\psi_0 \partial_\tau w = 0$ (this comes from the fact that $\partial_\tau \psi$ and $\partial_\tau \psi_0$ are zero), and $\psi_0 \partial_n w$ tends to 0 as r tends to 1 (this requires an asymptotic development of $\partial_n \psi - (\psi/\psi_0)\partial_n \psi_0$ as r tends to 1). Hence the sum of the three volume terms (7.34) is zero. We know that ψ_0 is radially increasing so that $\mathbf{x} \cdot \nabla \psi_0 > 0$, and all the terms of (7.34) are nonnegative. Hence the integrand is identically zero, which implies that w is equal to a constant of modulus 1.

Remark 7.6. The same kind of proof allows us to obtain uniqueness of solutions in ω_R with $w = 1$ on ∂B_R, since in this case, the boundary term (7.35) is zero and the others (7.36) have the same sign as the volume terms.

Remark 7.7. In the proof, we do not use that the energy F_0 is finite, but only that we can find a sequence R_n such that the energy density on $r = R_n$ times $|x|$ tends to 0. This still holds if we assume that $F(w, \omega_R) = o(\log R)$ as $R \to \infty$.

7.2.2 Existence of a solution to I_R

In this subsection, we prove that for problem (7.25)–(7.26), the minimum is achieved:

Lemma 7.8. *Let $R > 1$, $c > 0$, and $\delta > 0$. Then problem (7.25)–(7.26) has a minimizer ψ that satisfies*

$$\Delta \psi - 2i \frac{c}{1+\lambda} \partial_x \psi + (1-|\psi|^2)\psi = 0 \quad in \quad \omega_R, \tag{7.37}$$

for some $\lambda \geq 0$.

Proof: First, note that the minimization space, namely

$$X_R = \left\{ \psi \in H^1(\omega_R), \quad F_0\left(\frac{\psi}{\psi_0}, \omega_R\right) \leq \delta, \quad \psi \quad \text{satisfies} \quad (7.26) \right\},$$

is not empty. Indeed, $\psi_0 \in X_R$ since $F_0(1, \omega_R) = 0 \leq \delta$.

Next, we point out that $I_R > -\infty$. Let $w = \frac{\psi}{\psi_0}$. We have

$$
\begin{aligned}
|L(w,\omega_R)| &\leq K\left(\int_{\omega_R}\psi_0^2|w|^2\right)^{1/2}\left(\int_{\omega_R}\psi_0^2|\nabla w|^2\right)^{1/2}\\
&\leq K\sqrt{R}\left(\int_{\omega_R}\psi_0^2|\nabla w|^2\right)^{1/2}\left(\int_{\omega_R}\psi_0^4|w|^4\right)^{1/4}\\
&\leq K\sqrt{R}\sqrt{F_0(w,\omega_R)}\left(\int_{\omega_R}\psi_0^4(|w|^2-1)^2+R^2\right)^{1/4}\\
&\leq K\left(\sqrt{R}\,(F_0(w,\omega_R))^{3/4}+R\,(F_0(w,\omega_R))^{1/2}\right),
\end{aligned}
$$

which is bounded. Since $F_c = F_0 - cL$, this shows that $F_c(w,\omega_R)$ is bounded from below by some constant depending on R, but not on w.

Consider now a minimizing sequence of problem (7.25). This sequence is bounded in $H^1(\omega_R)$, so that we may extract a subsequence converging weakly in $H^1(\omega_R)$ and strongly in $L^p(\omega_R)$ for all $p < +\infty$. This allows to pass to the limit in the energy, and thus find a solution $\psi_{c,R}$ of (7.25).

We want to apply Theorem 9.2-2 of [46], to know that the solution $\psi_{c,R}$ of (7.25) satisfies the corresponding Euler–Lagrange equation, namely (7.37), and the Lagrange multiplier λ associated with the constraint is nonnegative. For this purpose, one needs to know that the constraints are qualified, that is, if there exists $\psi \in X_R$ such that $F_0\left(\frac{\psi}{\psi_0},\omega_R\right) = \delta$, then the derivative $F_0'\left(\frac{\psi}{\psi_0}\right)$ is not zero. This is a consequence of the fact that $F_0'(w) = -\mathrm{div}\,(\psi_0^2\nabla w) + \psi_0^4(|w|^2-1)w$, and that $w = 1$ is the unique solution of the equation such that $w\psi_0$ is in the space X_R, as pointed out in Remark 7.6.

7.2.3 Bounds on the solutions of I_R

In this section, we prove bounds on the minimizer $\psi_{c,R}$ of I_R (7.25)–(7.26).

Lemma 7.9. *Let $R > 1$, $\lambda \geq 0$, and $c > 0$. Let $\psi_{c,R}$ be a solution of (7.37) with boundary conditions (7.26). Let $w = \frac{\psi_{c,R}}{\psi_0}$. Then there exists a constant K independent of R, λ, and c such that*

(i) $\|\psi_{c,R}\|^2_{L^\infty(\omega_R)} \leq 1 + c^2$,
(ii) $\|\nabla\psi_{c,R}\|^2_{L^\infty(\omega_R)} \leq K(1+c^2)^3$,
(iii) $\|w\|^2_{L^\infty(\omega_R)} \leq K(1+c^2)^3$,
(iv) $\|\nabla w\|^2_{L^\infty(\omega_R)} \leq K(1+c^2)^4$.

Proof: Since $\lambda \geq 0$, we have $\frac{c}{1+\lambda} \leq c$, so that we may consider without loss of generality that $\lambda = 0$. Hence $\psi_{c,R}$ satisfies

$$\Delta\psi_{c,R} - 2ic\partial_x\psi_{c,R} + (1-|\psi_{c,R}|^2)\psi_{c,R} = 0.$$

Consider now $\eta(\mathbf{x}) = \psi_{c,R}(\mathbf{x})e^{-icx}$. This function satisfies

$$\Delta\eta + (1 + c^2 - |\eta|^2)\eta = 0.$$

Hence, setting $f = |\eta|^2$, we have

$$\Delta f + 2(1 + c^2 - f)f = 2|\nabla\eta|^2 \geq 0 \quad \text{in} \quad \omega_R.$$

Consider an interior maximum of f. At this point, $\Delta f \leq 0$, so that $f \leq 1 + c^2$. Since on $\partial\omega_R$, $f \leq 1 \leq 1 + c^2$, this shows that $f \leq 1 + c^2$. Since $|\psi_{c,R}|^2 = f$, we obtain (i).

Next, (ii) follows from the Gagliardo–Nirenberg inequality (see for instance [32]):

$$\|\nabla\eta\|_{L^\infty(\omega_R)} \leq K\left(\|\eta\|_{L^\infty(\omega_R)} + \|\Delta\eta\|_{L^\infty(\omega_R)}\right) \leq K(1 + c^2)^{3/2},$$

for some constant K independent of R.

We next prove (iii): a similar proof to that in Section 2.2 yields that for $r \in [1, 2]$, $|w| \leq \frac{\|\nabla\psi_{c,R}\|_{L^\infty}}{\psi_0(2)}$, and for $r \geq 2$, $|w(\mathbf{x})|^2 \leq \frac{1+c^2}{\psi_0(2)^2}$. This proves (iii).

We now turn to (iv), and use the same property of ψ_0, together with the identity $\nabla w = \frac{\nabla\psi_{c,R}}{\psi_0} - \frac{w\nabla\psi_0}{\psi_0}$, and (ii) and (iii). This shows that $|\nabla w(\mathbf{x})|^2 \leq \frac{K(1+c^2)}{(|\mathbf{x}|-1)^2}$ if $|\mathbf{x}| \leq 2$, and that $|\nabla w(\mathbf{x})|^2 \leq K(1+c^2)^3$ elsewhere. Next, we use Taylor expansions of $\psi_{c,R}$, $\partial_r\psi_{c,R}$, ψ_0, and ψ_0' near $r = 1$. This will prove the desired inequality for $1 \leq |\mathbf{x}| \leq 2$, concluding the proof. Indeed, consider first the tangential derivative: we have $\partial_\theta w = \frac{\partial_\theta\psi_{c,R}}{\psi_0} - \frac{\psi_{c,R}}{\psi_0^2}\partial_\theta\psi_0$. A proof similar to w bounded allows us to obtain

$$\left|\frac{1}{r}\partial_\theta w\right| \leq K(1 + c^2)^2 \quad \text{in} \quad B_2 \setminus \overline{B_1}.$$

Turning to the radial derivative, we have $\partial_r w = \frac{\partial_r\psi_{c,R}}{\psi_0} - \frac{\psi_{c,R}}{\psi_0^2}\partial_r\psi_0$. We claim that

$$\begin{cases} \psi_0 = \psi_0'(1)(r-1) + O(r-1)^2, \\ \psi_0' = \psi_0'(1) + O(r-1), \\ \psi_{c,R} = (r-1)\partial_r\psi_{c,R}(1,\theta) + (1+c^2)^2 O(r-1)^2, \\ \partial_r\psi_{c,R} = \partial_r\psi_{c,R}(1,\theta) + (1+c^2)^2 O(r-1), \end{cases} \tag{7.38}$$

where the terms $O(r-1)^k$ involve constants independent of R and c. Inserting this into the definition of $\partial_r w$, we find that $\partial_r w = \left(1 + \frac{(1+c^2)^2}{\psi_0'(1)}\right)O(1)$, where the $O(1)$ involves constants independent of R and c. This concludes the proof of (iv).

We next prove that $|w|$ cannot be far from 1 in some sense:

Lemma 7.10. *There exist $K > 0$ and $\delta_0 > 0$ depending only on the unique solution ψ_0 of (7.20)–(7.21) such that for any R sufficiently large, any $w \in W^{1,\infty}(\omega_R)$, and any*

$$\delta \leq \inf\left\{\delta_0, \frac{K}{\|\nabla w\|_\infty^{12}}\right\},$$

$F_0(w, \omega_R) \leq \delta$ implies that

$$\frac{1}{2} \leq |w| \leq \frac{3}{2}. \tag{7.39}$$

The proof is similar to [32]. We just need to take into account that near $r = 1$, the weight ψ_0 is small. The key point here is that this region is of small measure.

Proof: We prove only the lower bound, the same method applying to the case of the upper bound. Without loss of generality, we may assume that $\|\nabla w\|_{L^\infty} \geq 1$. Let

$$\alpha = \delta^{1/12}, \quad \eta = 32\delta^{1/3}.$$

If $\delta \leq \inf\{\delta_0, \frac{1}{(256\|\nabla w\|_\infty)^3}, (\frac{\psi_0(2)}{8\|\nabla w\|_\infty})^{12}\}$, where δ_0 depends only on ψ_0, the following inequalities hold:

$$\alpha \leq \psi_0'(1) = \|\nabla\psi_0\|_\infty, \;\; \eta \leq \frac{\alpha}{2\|\nabla\psi_0\|_\infty}, \quad \eta \leq \frac{1}{8\|\nabla w\|_\infty},$$
$$\alpha \leq \frac{\psi_0(2)}{8\|\nabla w\|_\infty}, \;\; \alpha \leq \frac{\psi_0(2)}{2}. \tag{7.40}$$

We argue by contradiction and assume that there exists $\mathbf{x}_0 \in \omega_R$ such that $|w(\mathbf{x}_0)| < \frac{1}{2}$.

1st case: $\psi_0(\mathbf{x}_0) > \alpha$. This implies that $|\mathbf{x}_0| - 1 \geq \frac{\alpha}{\psi_0'(1)} \geq \eta$. Hence, $B(\mathbf{x}_0, \eta) \subset \omega$. It may be that $B(\mathbf{x}_0, \eta) \not\subset \omega_R$, but at least $|B(\mathbf{x}_0, \eta) \cap \omega_R| \geq \frac{\pi}{3}\eta^2$. Let us compute $F_0(w, B(\mathbf{x}_0, \eta) \cap \omega_R)$. Using respectively the second and third inequalities of (7.40), one shows that $\psi_0 \geq \frac{\alpha}{2}$ and $|w| \leq \frac{3}{4}$ in $B(\mathbf{x}_0, \eta) \cap \omega_R$. Hence,

$$F_0(w, \omega_R) \geq \frac{1}{4}\int_{B(\mathbf{x}_0,\eta)\cap\omega_R} \psi_0^4(1 - |w|^2)^2 \geq \frac{1}{4}\frac{\pi}{3}\eta^2\frac{\alpha^4}{16}\frac{1}{16} = \frac{\pi}{3}\delta > \delta,$$

which is a contradiction to the hypothesis $F_0(w, \omega_R) \leq \delta$.

2nd case: $\psi_0(\mathbf{x}_0) \leq \alpha$. Since ψ_0 is radially symmetric and concave with respect to r, we then have $\frac{1}{|\mathbf{x}_0|-1}\psi_0(\mathbf{x}_0) \geq \psi_0(2)$; hence $|\mathbf{x}_0| - 1 \leq \frac{\alpha}{\psi_0(2)}$. Let $\mathbf{x}_1 = (1 + \frac{\alpha}{|\mathbf{x}_0|\psi_0(2)})\mathbf{x}_0$. Then, $|\mathbf{x}_1| = |\mathbf{x}_0| + \frac{\alpha}{\psi_0(2)} \geq 1 + \frac{\alpha}{\psi_0(2)}$ and $|\mathbf{x}_1| \leq R$ if $R \geq 2$. According to the fifth equation of (7.40), we thus have

$$\psi_0(\mathbf{x}_1) \geq (|\mathbf{x}_1| - 1)\psi_0(2) \geq \alpha, \quad \text{and} \quad |w(\mathbf{x}_1)| \leq |w(\mathbf{x}_0)| + \beta\|\nabla w\|_\infty \leq \frac{5}{8},$$

where we have used the fourth equation of (7.40). We thus come to a case similar to the first one, and the same computations give $\psi_0 \geq \frac{\alpha}{2}$ and $1 - |w| \geq \frac{1}{4}$ on $B(\mathbf{x}_1, \eta)$. Hence

$$F_0(w, \omega_R) \geq \frac{1}{4}\int_{B(\mathbf{x}_1,\eta)} \psi_0^4(1 - |w|^2)^2 \geq \frac{1}{4}\frac{\pi}{3}\eta^2\frac{\alpha^4}{16}\frac{1}{16} = \frac{\pi}{3}\delta > \delta,$$

which is here again a contradiction.

7.2.4 Estimating the momentum

In this subsection, we prove an estimate of the momentum L in terms of the energy F_0. This will allow us to show that the constraint in (7.25) is not active, and therefore that $\lambda = 0$ in (7.37).

Lemma 7.11. *Let $R \geq 2$, $c > 0$, and let $\psi \in H^1(\omega_R)$ satisfying (7.26) and such that $w = \frac{\psi}{\psi_0}$ satisfies (7.39). Then there exists a constant K independent of ψ, R, and c such that*

$$|L(w, \omega_R)| \leq K \left(F_0(w, \omega_R) + \sqrt{F_0(w, \omega_R)} \right). \tag{7.41}$$

Proof: Since w satisfies (7.39), we know that there exist $\rho, \phi \in H^1(\omega_R)$ such that $\rho \geq 1/2$ and

$$w = \rho e^{i(\phi + d\theta)},$$

where $d \in \mathbf{Z}$, and θ is the polar angle of $\mathbf{x}$. In addition, the fact that $\nabla w \in L^2(B_2^c)$ implies that d must be zero. Using the equality above in the definition of L, we obtain

$$L(w, \omega_R) = \int_{\omega_R} \frac{i}{2} \psi_0^2 \left(w \partial_x \overline{w} - \overline{w} \partial_x w \right) = \int_{\omega_R} \psi_0^2 \rho^2 \partial_x \phi.$$

Let $\alpha \in \left(\frac{1}{2}, 1 \right)$ (which will be made precise below), and consider separately the integral over $\{r < 1 + \alpha\} \cap \omega_R$ and over $\{r > 1 + \alpha\} \cap \omega_R$. We have

$$\begin{aligned} \left| \int_{1<r<1+\alpha} \psi_0^2 \rho^2 \partial_x \phi \right| &\leq \left(\int_{1<r<1+\alpha} \psi_0^2 |\nabla w|^2 \right)^{\frac{1}{2}} \left(\int_{1<r<1+\alpha} \psi_0^2 |w|^2 \right)^{\frac{1}{2}} \\ &\leq K \sqrt{F_0(w)} \left(\int_{1<r<1+\alpha} \psi_0^2 \right)^{\frac{1}{2}} \\ &\leq K \|\nabla \psi_0\|_\infty \alpha \sqrt{F_0(w)} \left((1+\alpha)^2 - 1 \right)^{\frac{1}{2}} \\ &\leq K \alpha^{3/2} \sqrt{F_0(w)}, \end{aligned} \tag{7.42}$$

where K depends only on ψ_0.

Turning to the integral over the set $\{r > 1 + \alpha\}$, we have

$$\begin{aligned} \int_{1+\alpha<r<R} \psi_0^2 \rho^2 \partial_x \phi = &\int_{1+\alpha<r<R} \psi_0^2 (\rho^2 - 1) \partial_x \phi + \int_{1+\alpha<r<R} (\psi_0^2 - 1) \partial_x \phi \\ &+ \int_{1+\alpha<r<R} \partial_x \phi. \end{aligned} \tag{7.43}$$

We consider separately the three terms above:

$$\begin{aligned} \left| \int_{1+\alpha<r<R} \psi_0^2 (\rho^2 - 1) \partial_x \phi \right| &\leq \left(\int_{1+\alpha<r<R} \psi_0^2 (\rho^2 - 1)^2 \right)^{\frac{1}{2}} \left(\int_{1+\alpha<r<R} \psi_0^2 \partial_x \phi^2 \right)^{\frac{1}{2}} \\ &\leq \frac{K}{\alpha} \left(\int_{\omega_R} \psi_0^4 (\rho^2 - 1)^2 \right)^{\frac{1}{2}} \left(\int_{\omega_R} \psi_0^2 \rho^2 |\nabla \phi|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{K}{\alpha} F_0(w). \end{aligned} \tag{7.44}$$

Here we have used the fact that $\rho > 1/2$ and there exists a constant K independent of α such that in $\{r \geq 1+\alpha\}$, we have $\psi_0 \geq K\alpha$. The second term is dealt with in a similar way:

$$\left| \int_{1+\alpha<r<R} (\psi_0^2 - 1)\partial_x \phi \right| \leq 2 \left(\int_{1+\alpha<r<R} \frac{(\psi_0^2-1)^2}{\psi_0^2} \right)^{\frac{1}{2}} \left(\int_{\omega_R} \psi_0^2 \rho^2 \partial_x \phi^2 \right)^{\frac{1}{2}}$$
$$\leq \frac{K}{\alpha} \sqrt{F_0(w)}. \tag{7.45}$$

This is due to the fact that ψ_0 is bounded below and $E_0(\psi_0)$ is finite. Finally, we integrate by parts the last term and get (recall that $\phi = 0$ on $r = R$)

$$\left| \int_{1+\alpha<r<R} \partial_x \phi \right| = \left| \int_{r=1+\alpha} \phi n_x \right| = \left| \int_{r=1+\alpha} \left(\phi - \inf_{r=1+\alpha} \phi \right) n_x \right|$$
$$\leq \frac{K}{\alpha} \left(\int_{r=1+\alpha} \psi_0^2 \rho^2 |\nabla \phi|^2 \right)^{\frac{1}{2}}. \tag{7.46}$$

We next point out that since $\int_{1<r<2} \psi_0^2 \rho^2 |\nabla \phi|^2 \leq 2F_0(w)$, there exists $\alpha \in \left(\frac{1}{2}, 1\right)$ such that

$$\int_{r=1+\alpha} \psi_0^2 \rho^2 |\nabla \phi|^2 \leq 4F_0(w).$$

This inequality, together with (7.42), (7.43), (7.44), (7.45), and (7.46), implies (7.41).

The results above allow us to show that the Lagrange multiplier λ is in fact zero.

Proposition 7.12. *There exist $\delta_1 > 0$ and $K > 0$ such that for all $R \geq 2$, $\delta \leq \delta_1$, and $c \in (0, K\sqrt{\delta})$, any minimizer $\psi_{c,R}$ of (7.25) with boundary conditions (7.26) satisfies $F_0\left(\frac{\psi_{c,R}}{\psi_0}\right) < \delta$. In addition, $\psi_{c,R}$ is a solution of*

$$\Delta\psi - 2ic\partial_x\psi + (1-|\psi|^2)\psi = 0 \quad in \quad \omega_R. \tag{7.47}$$

Proof: Consider $c, \delta \leq 1$, and let $\psi_{c,R}$ be a minimizer of (7.25). Applying Lemma 7.8 and then Lemma 7.9, we find that there is a constant $K_1 > 0$ independent of R, c, and δ such that if $w = \frac{\psi_{c,R}}{\psi_0}$, then $\|\nabla w\|_\infty \leq K_1$. Hence, applying Lemma 7.10, we find that there exists some $\delta_1 > 0$ independent of R, c, and δ such that if $\delta \leq \delta_1$, $F_0(w, \omega_R) \leq \delta$ implies that w satisfies (7.39).

We now apply Lemma 7.11 and find that for some constant K_2 independent of R, c, and δ, we have $|L(w, \omega_R)| \leq K_2(F_0(w, \omega_R) + \sqrt{F_0(w, \omega_R)}) \leq 2K_2\sqrt{F_0(w, \omega_R)}$. We want to prove that the constraint in (7.25) is not active, that is, the minimizer cannot satisfy $F_0(w, \omega_R) = \delta$. Assume that this is the case. The estimate on the momentum implies that

$$F_c(w, \omega_R) \geq \delta - 2cK_2\sqrt{\delta}.$$

Now, $\psi = \psi_0$ is a test function for problem (7.25). Hence, we must have $F_c(w) \leq F_c(1) = 0$. Thus,

$$0 \geq \delta - 2cK_2\sqrt{\delta}.$$

If $c < \frac{1}{2K_2}\sqrt{\delta}$, this is a contradiction. Hence, for any minimizer $\psi_{c,R}$, $F_0\left(\frac{\psi_{c,R}}{\psi_0}, \omega_R\right) < \delta$. The constraint in (7.25) is not active, and the corresponding Lagrange multiplier must be zero. Hence, $\psi_{c,R}$ satisfies (7.37) with $\lambda = 0$, namely (7.47).

7.2.5 Proof of Theorem 7.4

We now conclude the proof of Theorem 7.4. We apply Proposition 7.12, and find that for some $c_0 = K\sqrt{\delta_1}$, there exists a solution $\psi_{c,R}$ of (7.47) with boundary conditions (7.26). In addition, this function satisfies

$$F_0\left(\frac{\psi_{c,R}}{\psi_0}, \omega_R\right) < \delta, \quad \|\psi_{c,R}\|_{W^{1,\infty}(\omega_R)} \leq K, \tag{7.48}$$

for some constants K and δ independent of R. Thus we can extract a subsequence with weak convergence in H^1_{loc} and strong convergence in L^4_{loc}. At the limit $R \to +\infty$, it yields a solution ψ_c of (7.2)–(7.3) such that $F_0(\psi_c/\psi_0) \leq \delta$, ψ_c is bounded in $W^{1,\infty}$, $\frac{1}{2} \leq \left|\frac{\psi_c}{\psi_0}\right| \leq \frac{3}{2}$. It implies in particular that the solution is vortex-free. Using the equation for ψ_c, we find that as $c \to 0$, ψ_c converges to ψ_0 (up to multiplication by a constant of modulus one) in L^∞_{loc}. To show that we have convergence in $L^\infty(\omega)$, we point out that ψ_c converges to 1 at infinity, uniformly with respect to $c \to 0$. This is proved in Lemma 7.13 below.

The uniqueness of the solutions of (7.2)–(7.3) with finite energy will be proved only in the 3D case, since the arguments are very similar.

7.2.6 Limit at infinity

Lemma 7.13. *Let $M > 0$, and let ψ_c be a solution of (7.2)–(7.3) such that $E_0(\psi_c) \leq M$. Then, up to multiplication by a constant of modulus one,*

$$\lim_{|\mathbf{x}|\to\infty} \psi_c(\mathbf{x}) = 1, \tag{7.49}$$

uniformly with respect to $c \longrightarrow 0$.

Proof: We follow the proof of [72, 74] , in which such a property is established for the same equation in $\mathbf{R}^3$. The proof of [72, 74] does not work in dimension 2, but with a more precise estimate on the decay of the energy, we are able to adapt it.

Step 1: Limit of $|\psi_c|$: $\lim_{|\mathbf{x}|\to\infty} |\psi_c(\mathbf{x})| = 1$, *uniformly with respect to c.*

Proof: This property may be directly derived from the upper bound on the gradient and the fact that the energy is finite. We refer to [41] for the details. Here, the additional property we need is that the limit is uniform with respect to $c \to 0$. We argue by contradiction and assume that there exists $\varepsilon > 0$, a sequence $c_n \to 0$, and a sequence $\mathbf{x}_n$ such that $|\mathbf{x}_n| \to \infty$ and

$$\big||\psi_{c_n}(\mathbf{x}_n)| - 1\big| \geq \varepsilon.$$

Consider now the function $\tilde{\psi}_n = \psi_{c_n}(\cdot + \mathbf{x}_n)$. It is bounded in L^∞, and satisfies (7.2) and $E_0(\psi_n) \leq M$. Hence, passing to the limit, we find a solution of (7.2) in $\mathbf{R}^2$ with finite energy and 0 degree. But this must be a constant of modulus one, according to [41].

Step 2: Decay of the energy: *Let* $e(\psi) = \frac{1}{2}|\nabla\psi|^2 + \frac{1}{4}(1 - |\psi|^2)^2$. *There exist* $K > 0$, $\alpha > 1$, *and* $R_0 > 1$ *independent of* c *such that for* $R > R_0$,

$$\int_{B_R^c} e(\psi_c) \leq \frac{K}{R^\alpha}. \tag{7.50}$$

The following argument is a slight improvement of the proof of Proposition 28 of [72] (see also [34]), to which we refer for details. The extra information we need here is that in the decay, $\alpha > 1$.

Proof: Let ε be a positive constant, to be made precise later on. We consider R_0 large enough so that $1 - \varepsilon \leq |\psi_c| \leq 1 + \varepsilon$ for $r > R_0$. This R_0 may be chosen independent of c. Moreover, as pointed out in the proof of Lemma 7.11, we know that there exist $\rho > 0$ and ϕ such that $\psi_c = \rho e^{i\phi}$. Inserting this decomposition in (7.2), we obtain

$$\begin{cases} \Delta\rho - \rho|\nabla\phi|^2 + 2c\rho\partial_x\phi = \rho(\rho^2 - 1), \\ \text{div}\,(\rho^2\nabla\theta) = c\partial_x(\rho^2). \end{cases} \tag{7.51}$$

Let $\phi_R = \frac{1}{2\pi R^2}\int_{S_R}\phi$, where S_R is the sphere of radius R. We multiply the second equation of (7.51) by $\phi - \phi_R$ and integrate, and then multiply the first equation by $\rho^2 - 1$ and integrate over B_R^c. Adding the results, we obtain

$$\begin{aligned} \int_{B_R^c} e(\psi_c) = {} & \frac{1}{4}\int_{B_R^c} \rho(1-\rho^2)|\nabla\phi|^2 + \int_{B_R^c}(1-\rho)\left(\frac{|\nabla\rho|^2}{2} + \frac{(1-\rho^2)^2}{4}\right) \\ & + c\int_{B_R^c}\rho(\rho^2-1)\partial_x\phi + \frac{c}{2}\int_{B_R^c}(1-\rho)(\rho^2-1)\partial_x\phi \\ & - \frac{1}{4}\int_{S_R}\partial_n\rho(\rho^2-1) - \frac{1}{2}\int_{S_R}\rho^2\partial_n\phi(\phi-\phi_R) \\ & + \frac{c}{2}\int_{S_R}(\phi-\phi_R)(\rho^2-1)n_x. \end{aligned} \tag{7.52}$$

Estimating each term of the right-hand side of (7.52) separately and using the Poincaré inequality on S_R, one easily gets

$$\begin{aligned} \int_{B_R^c} e(\psi_c) \leq {} & (c\sqrt{2} + 3\varepsilon)\int_{B_R^c} e(\psi_c) \\ & + R\left(\frac{\sqrt{1+c^2}}{2}(1+\varepsilon) + c\varepsilon + \frac{1}{2\sqrt{2}R}\right)\int_{S_R} e(\psi_c). \end{aligned}$$

Let $J(R) = \int_{B_R^c} e(\psi_c)$. Choosing c and ε small enough ($c < \frac{\sqrt{2}-1}{4}$ is sufficient here), we find that there exists $A < 1$ such that $J(R) \leq -ARJ'(R)$. This implies that $J(R) \leq KR^{-\frac{1}{A}}$, and yields (7.50).

The key point that we have checked here is that indeed $1/A > 1$. In the sequel, we set, for any $r \geq 1$,

$$\psi_c^r(\xi) = \psi_c(r\xi), \quad \xi \in S_1.$$

Step 3: Existence of a limit

$$\exists \psi_c^\infty \in L^2(S_1) \quad \textit{such that} \quad \psi_c^r \underset{r\to\infty}{\longrightarrow} \psi_c^\infty \quad \textit{in} \quad L^2(S_1), \tag{7.53}$$

uniformly with respect to $c \to 0$.

Proof: We first point out that if $f(r) = \int_{S_r} |\nabla\psi_c|^2$, we know that $\|f\|_{L^1(1,+\infty)}$ is bounded independently of c, and that $\int_R^{+\infty} f(r)dr \leq \frac{K}{R^\alpha}$, with $\alpha > 1$ and $K > 0$ independent of c. This clearly implies that $\|rf\|_{L^1(1,+\infty)}$ is bounded independently of c, and thus that $\left\| |\mathbf{x}||\nabla\psi_c|^2 \right\|_{L^1(\omega)}$ inherits this property.

Moreover, we have

$$\begin{aligned}
\int_{S_1} |\psi_c^r - \psi_c^{r'}|^2 &\leq \int_{S_1} \left| \int_r^{r'} \partial_s \psi_c^s(\xi) \right|^2 d\xi \\
&\leq \int_{S_1} \Big(\int_r^{r'} \frac{ds}{s^2} \Big) \Big(\int_r^{r'} |\nabla\psi_c|^2(s\xi) s^2 ds \Big) d\xi \\
&\leq \left(\frac{1}{r} - \frac{1}{r'} \right) \int_{B_{r'}\setminus B_r} |\mathbf{x}||\nabla\psi_c|^2 d\mathbf{x},
\end{aligned}$$

which proves that ψ_r is a Cauchy sequence in $L^2(S_1)$, uniformly with respect to $c \to 0$, from which we deduce (7.53).

Step 4: ψ_c^∞ is constant:

Proof: We know that for a sequence R_n going to infinity, $\int_{S_{R_n}} |\mathbf{x}||\nabla\psi_c|^2$ converges to zero as n tends to infinity. Thus, using the inequality $|\nabla\psi_c(r\xi)|^2 \geq \frac{1}{r^2}|\nabla^{S_1}\psi_c^r|^2$, we have

$$\int_{S_1} |\nabla^{S_1}\psi_c^r|^2 \leq \int_{S_1} |\nabla\psi(r\xi)|^2 r^2 d\xi = \int_{S_r} |\mathbf{x}||\nabla\psi_c|^2,$$

which proves that for a sequence R_n going to infinity, $\nabla^{S_1}\psi_c^{R_n}$ converges to 0 as n goes to infinity. Hence, $\nabla^{S^1}\psi_c^\infty = 0$.

We now conclude by pointing out that $H^1(S_1)$ is embedded into $L^\infty(S_1)$, and we thus have convergence of ψ_c^r to ψ_c^∞ in $L^\infty(S_1)$, which proves that $\lim_{|\mathbf{x}|\to\infty} \psi_c(\mathbf{x}) = \psi_c^\infty$. Using the fact that $E_0(\psi_c) < +\infty$, we find that the constant ψ_c^∞ is a constant of modulus one, concluding the proof.

7.3 Proof of Theorem 7.2

Theorem 7.14. *There exists a unique nontrivial nonnegative solution u_0 of*

$$\Delta u_0 + (z - |u_0|^2)u_0 = 0 \quad in \quad \Omega = (\mathbf{R}^2 \setminus \overline{B}_1) \times (0, 1), \tag{7.54}$$

with boundary condition (7.15) and $c = 0$, namely

$$u = 0 \text{ on } \{z = 0\} \text{ and } \{r = 1\}, \quad u = \psi_0 \text{ on } \{z = 1\}, \tag{7.55}$$

where ψ_0 is defined in Theorem 7.3. The solution u_0 depends on r and z, is increasing in r and in z. Moreover, $(u_0 - p)$ tends to 0 exponentially fast when r is large, where p is the solution of (7.17), u_0 is the unique solution of (7.54)–(7.55) in

$$Y = \left\{ u \in H^1_{loc}(\Omega, \mathbf{C}),\ \mathcal{F}_0\left(\frac{u}{u_0}\right) < \infty \right\},$$

where $\mathcal{F}_0$ is defined by (7.19).

Remark 7.15. Let us point out that the invariance with respect to multiplication by a constant of modulus one, which appears in Theorem 7.3, is lost in the three-dimensional case due to the boundary condition on $\{z = 1\}$, which fixes the corresponding phase.

Theorem 7.16. *There exists $c_0 > 0$ such that for all $c \in (0, c_0)$, problem (7.14)–(7.15) has a vortex-free solution u_c. Moreover, as c tends to 0, if the upper boundary condition ψ_c in (7.15) is the one tending to ψ_0, then u_c tends to u_0 in $L^\infty(\Omega)$. For all M, there exists c_1 such that for $c < c_1$, u_c is the unique solution with $\mathcal{F}_0(u_c/u_0) < M$:*

$$\mathcal{F}_c(w, \Omega_R) = \mathcal{F}_0(w, \Omega_R) - c\mathcal{L}(w, \Omega_R), \tag{7.56}$$

where

$$\mathcal{L}(w, \Omega_R) = \int_{\Omega_R} u_0^2 (iw, \partial_x w). \tag{7.57}$$

The corresponding minimization problem is

$$\mathcal{I}_R = \inf \left\{ \mathcal{F}_c\left(\frac{u}{u_0}, \omega_R\right), \quad u \in H^1(\Omega_R), \quad \mathcal{F}_0\left(\frac{u}{u_0}, \omega_R\right) \le \delta \right\}, \tag{7.58}$$

with boundary conditions

$$u = 0 \text{ on } \{r = 1\} \text{ and } \{z = 0\},\ u = u_0 \text{ on } \{r = R\},\ u = \psi_{c,R} \text{ on } \{z = 1\}, \tag{7.59}$$

where u_0 is defined in Theorem 7.14 and $\psi_{c,R}$ is the 2D solution constructed above.

7.3.1 Proof of Theorem 7.14

Existence of u_0

In this subsection, we prove the existence of a solution of (7.54)–(7.55). As in the 2D case, we first construct real-valued solutions in $\Omega_R = \omega_R \times (0, 1)$. We want to solve (7.54) with boundary conditions (7.15) and $u(R, z) = \psi_0(R)p(z)$: 0 is a subsolution and ψ_0 is a supersolution; hence there is a real positive solution u_R in between. Using the moving plane and sliding methods [28, 29], we can prove that u_R depends on r and z, and is increasing in r and in z. In particular, $\|u_R\|_\infty \leq \|\psi_0\|_\infty \leq 1$. Classical elliptic estimates yield uniform bounds that allow one to pass to the limit in R and obtain a positive real solution u_0 of (7.54)–(7.55) in Ω. Moreover, u_0 is also increasing in r and in z.

Properties of u_0

Let u_0 be the solution obtained above. We prove here that u_0 is the unique nonnegative nontrivial solution of (7.54)–(7.55) and that $u_0 - p$ goes to 0 exponentially fast at infinity. The proof is divided into several steps.

Step 5: For all $r_0, \gamma > 0$, there exists $\beta > 0$ such that $\partial u_0/\partial n \geq \beta$ on $\{r = 1\} \cap \{z \geq \gamma\}$ and on $\{z = 0\} \cap \{r \geq r_0\}$. *Moreover, there exists K such that for $r \geq r_0$, $u_0 \geq Kz$.*

Note that we have to avoid the corner $r = 1$, $z = 0$ where the normal derivatives go to zero.

Proof: On $\{r = 1\}$, this is a consequence of the Hopf lemma. On $z = 0$, let us prove it by contradiction and assume that there is a sequence $\mathbf{x}_n$ on $\{z = 0\}$ such that $|\partial u_0/\partial n(\mathbf{x}_n)|$ tends to 0. Applying the Hopf lemma, we find that necessarily, $|\mathbf{x}_n|$ tends to infinity. Let $u_n(\mathbf{x}) = u_0(\mathbf{x} + \mathbf{x}_n)$. Since u_n is bounded in L^∞ and in H^2_{loc}, it converges uniformly on every compact subset to $\overline{u}$, which is a solution of

$$\Delta\overline{u} + \overline{u}(z - \overline{u}^2) = 0 \text{ in } \mathbf{R}^2 \times (0, 1),\ \overline{u} \geq 0, \tag{7.60}$$

$$\overline{u} = 0 \text{ on } \{z = 0\},\ \overline{u} = 1 \text{ on } \{z = 1\}. \tag{7.61}$$

The boundary condition at $\{z = 1\}$ comes from the limit of ψ_0 at infinity. We also have at the limit $\frac{\partial \overline{u}}{\partial n}(0) = 0$, which provides a contradiction with the Hopf lemma.

The last statement comes from the lower bound on $\partial u_0/\partial n$ for $z < \gamma$ and the fact that for $z > \gamma$, $\overline{u}$ cannot vanish; hence u_0 is bounded below.

Step 6: Let $\mathbf{x}_0 \in \{r = 1\} \cap \{z = 0\}$, and let ξ be a direction at $\mathbf{x}_0$ that enters Ω nontangentially. Then $\frac{\partial^2 u_0}{\partial \xi^2}(\mathbf{x}_0) > 0$. *In particular, for all γ and r_0, there exists K such that $u \geq Kz^2$ in $\{z < \gamma\} \cap \{r < r_0\}$.*

Proof: The fact that $u_0 = 0$ on $\{z = 0\}$ and on $\partial B_1 \times (0, 1)$ implies that $\frac{\partial u_0}{\partial \xi}(\mathbf{x}_0) = 0$. The property on the second derivative thus follows from the Serrin corner lemma (see [145], lemma 1). This implies the bound from below for u_0.

Step 7: Nondegeneracy of u_0: *Let $\phi \in L^\infty(\Omega)$ be a complex-valued solution of*

$$\Delta\phi + (z - u_0^2)\phi - u_0^2(\phi + \overline{\phi}) = 0 \text{ in } \Omega,\ \phi = 0 \text{ on } \{z = 0\},\ \{z = 1\},\ \{r = 1\}. \tag{7.62}$$

Then $\phi \equiv 0$.

Proof: Let us separate ϕ into real and imaginary parts, a and b. We have

$$\Delta a + (z - 3u_0^2)a = 0, \quad \Delta b + (z - u_0^2)b = 0,$$

with homogeneous boundary conditions. We use a proof adapted from [29] to get that b is zero. Consider the function $w = \frac{b}{u_0}$. It satisfies div $(u_0^2\nabla w) = 0$ in Ω, and vanishes on $\{z = 1\}$ and $\{r = 1\}$. We claim that w is bounded in $\overline{\Omega}$: for $z > \gamma$ and $r > r_0$, this comes from step 1, since u_0 is bounded below; near $z = 0$ and $r = 1$, the proof is similar to that in Section 2.2 and uses the bound from below of $\partial u_0/\partial n$ derived in step 1.

We are going to use a cutoff function ξ independent of z, defined by

$$\xi = 1 \text{ for } r \le R, \quad \xi = 0 \text{ for } r \ge 2R, \quad \xi = 1 - \frac{r}{2R} \text{ for } R \le r \le 2R. \tag{7.63}$$

Multiplying div $(u_0^2\nabla w) = 0$ by $w\xi^2$ and integrating, we have

$$\begin{aligned}\int_\Omega \xi^2 u_0^2|\nabla w|^2 &\le 2\left|\int_\Omega u_0^2\xi w\nabla\xi\cdot\nabla w\right|\\ &\le 2\left(\int_{\Omega\cap\{R<r<2R\}} u_0^2\xi^2|\nabla w|^2\right)^{1/2}\left(\int_\Omega u_0^2 w^2|\nabla\xi|^2\right)^{1/2}.\\ &\le C\left(\int_{\Omega\cap\{R<r<2R\}} u_0^2\xi^2|\nabla w|^2\right)^{1/2},\end{aligned}$$

which implies that $\int_\Omega \xi^2 u_0^2|\nabla w|^2 < +\infty$, and in turn that $\int u_0^2|\nabla w|^2 = 0$. Hence, $\nabla w = 0$, so that $b = \gamma u_0$, for some constant γ. But the boundary condition on $\{z = 1\}$ implies that $\gamma = 0$, so that $b = 0$.

Next, we prove that $a = 0$: $w = \frac{a}{u_0}$ satisfies div $(u_0^2\nabla w) - 2u_0^4 w = 0$. Hence, the same proof as above applies to this case, and yields

$$\int_\Omega \xi^2 u_0^2|\nabla w|^2 + 2\int_O u_0^4\xi^2 w^2 \le C\left(\int_{\Omega\cap\{R<r<2R\}} u_0^2\xi^2|\nabla w|^2\right)^{1/2}$$

showing that $\int u_0^2|\nabla w|^2 = 0$ and $a = 0$.

Step 8: Uniqueness of the real nonnegative solution.

Proof: Let u_0 be the solution obtained above, and consider a nonnegative solution u of (7.54)–(7.55). We define $w = \frac{u}{u_0}$ and get that div $(u_0^2\nabla w) - u_0^4 w(w^2 - 1) = 0$, with $w = 1$ on $\{z = 1\}$ and w is bounded. Thus, multiplying this equation by $\xi^2(w - 1)$, with ξ defined by (7.63), and using the same argument as above, we prove that w is a constant, and hence $w = 1$.

Remark 7.17. Similar proofs allow us to get that there is a unique solution of

$$\Delta u + u(z - |u|^2) = 0 \text{ in } \mathbf{R}^2 \times (0,1), \tag{7.64}$$

$$u = 0 \text{ on } \{z = 0\}, \ u = 1 \text{ on } \{z = 1\}, \tag{7.65}$$

which is also nondegenerate. This solution is $p(z)$, the solution of (7.17).

Step 9: Behaviour at ∞*:* $u_0 - p(z)$ *tends to* 0 *exponentially fast as* r *tends to* ∞*, uniformly in* z*.*

Proof: We first show that u_0 tends to $p(z)$ as r tends to infinity, uniformly in z: assume by contradiction that u_0 does not tend to $p(z)$ as r tends to ∞. Then, one can find a sequence $\mathbf{x}_n = (x_n, y_n, z_n)$ in Ω such that $|u_0(\mathbf{x}_n) - p(z_n)| \geq \varepsilon > 0$. Let $u_n(\mathbf{x}) = u_0(\mathbf{x} + \mathbf{x}'_n)$, where $\mathbf{x}'_n = (x_n, y_n, 0)$. Since u_0 is bounded, we can pass to the limit in n and find that u_n converges uniformly on every compact subset to $\overline{u}$, which is a solution of (7.60)–(7.61), with $|\overline{u}(0, \overline{z}) - p(\overline{z})| \geq \varepsilon$, where $\overline{z} = \lim z_n$. This provides a contradiction to Remark 7.17, since the only solution of (7.60)–(7.61) is $p(z)$.

We are now in position to prove that there exist some constants $K > 0$ and $\alpha > 0$ such that

$$|u_0(\mathbf{x}) - p(z)| \leq K e^{-\alpha r}, \tag{7.66}$$

where $r = \sqrt{x^2 + y^2}$. For this purpose, let us first define

$$M(R) = \sup_{r \geq R} |u_0(\mathbf{x}) - p(z)|.$$

Then, (7.66) is clearly equivalent to the following statement:

$$\exists R > 0, \quad \exists \gamma \in (0,1) \quad \text{s.t.} \quad \forall T \geq 1, \quad M(R+T) \leq \gamma M(T). \tag{7.67}$$

We argue by contradiction, and assume that there exist sequences R_n, γ_n, T_n satisfying the following:

$$\begin{cases} R_n \longrightarrow +\infty, \\ \gamma_n \longrightarrow 1, \\ T_n \geq 1, \end{cases} \qquad \text{and} \qquad M(R_n + T_n) > \gamma_n M(T_n).$$

Thus, one can find $\mathbf{x}_n \in \Omega$ such that $r_n = \sqrt{x_n^2 + y_n^2} \geq R_n + T_n$,

$$\frac{|u_0(\mathbf{x}_n) - p(z_n)|}{M(R_n + T_n)} \longrightarrow 1,$$

and $|u_0(\mathbf{x}_n) - p(z_n)| > \gamma_n M(T_n)$. We define the function f_n by

$$f_n(\mathbf{x}) = \frac{u_0(x + x_n, y + y_n, z) - p(z)}{M(R_n + T_n)}.$$

This function is bounded in $B_{R_n} \times (0,1)$, so we may extract a subsequence and pass to the limit in the equation. Since we already know that u_0 converges to $p(z)$ at infinity. This equation reads

$$\Delta f + (z - 3p^2) f = 0 \quad \text{in} \quad \mathbf{R}^2 \times (0,1), \tag{7.68}$$

with $|f(0, z_\infty)| = 1 \geq \|f\|_\infty$, where $z_\infty = \lim z_n$. In addition, f vanishes on $\{z = 0\}$ and $\{z = 1\}$. Remark 7.17 implies that f is zero. This is a contradiction.

Remark 7.18. A similar proof allows us to get that u_0/p tends to 1 exponentially fast as r goes to ∞, and in particular $(u_0 - p)/p$ is in $L^2(\Omega)$.

Uniqueness of u_0

In this section, we prove that any solution of (7.54)–(7.55) of finite energy (i.e., such that $\mathcal{F}_0(u/u_0) < \infty$) is in fact equal to u_0.

Let u be a solution of (7.54)–(7.55) with $\mathcal{F}(u/u_0)$ finite. Let $w = u/u_0$. Then w is a solution of

$$\operatorname{div}\left(u_0^2 \nabla w\right) + u_0^4\left(1 - |w|^2\right) w = 0 \text{ in } \Omega. \tag{7.69}$$

The boundary conditions are $w = 1$ on $z = 1$. Moreover, w is bounded. The proof of this fact is similar to the 2D case close to the obstacle; when z is close to 0, it uses step 5 of Section 7.3.1 and the bound below on u_0 by Kz far away from the obstacle and by Kz^2 close to the obstacle. The last estimate is a consequence of step 6 of Section 7.3.1.

The key tool is to use the Pohozaev identity as in the 2D case, but here we multiply by $\mathbf{x}/|\mathbf{x}| \cdot \nabla \overline{w}$ instead of just $\mathbf{x} \cdot \nabla \overline{w}$, integrate over Ω_R, and add the conjugate:

$$-\int_{\Omega_R} u_0^2 \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla(|\nabla w|^2) + 2\frac{u_0^2}{|\mathbf{x}|}|\nabla w|^2 + u_0^4 \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla\left(\frac{(1-|w|^2)^2}{2}\right) \tag{7.70}$$

$$+\int_{\Omega_R} 2u_0^2 \frac{|\mathbf{x} \cdot \nabla w|^2}{|\mathbf{x}|^3} + \int_{\partial\Omega_R} u_0^2 \frac{\partial w}{\partial n} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla \overline{w} + u_0^2 \frac{\partial \overline{w}}{\partial n} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla w = 0. \tag{7.71}$$

We integrate by parts the $\mathbf{x} \cdot \nabla w$ terms and obtain

$$\begin{aligned} &\int_{\Omega_R} \left(\frac{u_0^4}{|\mathbf{x}|} + 2u_0^3 \frac{\mathbf{x} \cdot \nabla u_0}{|\mathbf{x}|}\right)(1-|w|^2)^2 + 2u_0^2 \frac{|\mathbf{x} \cdot \nabla w|^2}{|\mathbf{x}|^3} + 2\frac{u_0 \mathbf{x} \cdot \nabla u_0}{|\mathbf{x}|}|\nabla w|^2 \\ &= \int_{\partial\Omega_R} \frac{1}{2} u_0^4 (1-|w|^2)^2 \frac{\mathbf{x}}{|\mathbf{x}|} \cdot n - u_0^2 \frac{\partial w}{\partial n} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla \overline{w} - u_0^2 \frac{\partial \overline{w}}{\partial n} \frac{\mathbf{x}}{|\mathbf{x}|} \cdot \nabla w \qquad (7.72) \\ &\quad + \int_{\partial\Omega_R} u_0^2 \frac{\mathbf{x}}{|\mathbf{x}|} \cdot n |\nabla w|^2. \end{aligned}$$

We are going to prove that the sum of the boundary terms is nonpositive as R tends to infinity. On $\{z = 0\}$ and $\{r = 1\}$, the same reasoning as in the 2D case allows to get that $u_0 \nabla w = 0$. Moreover, $u_0^2(1 - |w|^2) = u_0^2 - u^2 = 0$, so that the boundary terms are zero. On $\{z = 1\}$, $w = 1$, so the derivative is only in the normal direction and the boundary terms are equal to

$$-\int_{\{z=1\}\cap\partial\Omega_R} u_0^2 \left|\frac{\partial w}{\partial z}\right|^2,$$

which is negative. On $\{r = R\}$, the terms tend to 0 for a suitable sequence R_n tending to infinity because the energy $\mathcal{F}_0(w)$ is finite. In total, the sum of the volume terms is nonpositive as R_n tends to infinity. Since $\mathbf{x} \cdot \nabla u_0$ is positive, we find that each term is zero. This and the boundary condition on $\{z = 1\}$ yield that $w \equiv 1$.

Remark 7.19. The uniqueness result is also true for solutions in Ω_R with outer boundary condition $w = 1$ on $\{r = R\}$. Indeed, on $\{r = R\}$, $w = 1$ and the gradient of w is only in the normal direction, so that the boundary term is negative.

Remark 7.20. Let us point out that the power of $|\mathbf{x}|$ we use in the Pohozaev identity is linked to the dimension: indeed, the starting point of the method is the following formula, obtained by multiplying $-\Delta u$ by $|\mathbf{x}|^\alpha \mathbf{x} \cdot \nabla u$ and integrating by parts in $\mathcal{D}$, a domain of $\mathbf{R}^d$:

$$\begin{aligned}\int_{\mathcal{D}} (-\Delta u)\, |\mathbf{x}|^\alpha \mathbf{x} \cdot \nabla u = &\int_{\mathcal{D}} \alpha |x|^{\alpha-2} (\mathbf{x} \cdot \nabla u)^2 + \left(1 - \frac{\alpha + d}{2}\right) |\mathbf{x}|^\alpha |\nabla u|^2 \\ &+ \int_{\partial\mathcal{D}} \frac{1}{2} |\mathbf{x}|^\alpha \mathbf{x} \cdot n |\nabla u|^2 - \frac{\partial u}{\partial n} |\mathbf{x}|^\alpha \mathbf{x} \cdot \nabla u.\end{aligned}$$

Hence, in order to cancel one of the volume terms, we need to choose $\alpha = -d + 2$.

Remark 7.21. A similar proof allows us to obtain that $p(z)$ is the unique solution of (7.60)–(7.61) among complex-valued functions such that $\mathcal{F}_p(u/p)$ is finite, where $\mathcal{F}_p$ is defined similarly to $\mathcal{F}_0$, with u_0 replaced by p.

7.3.2 Proof of Theorem 7.16

We are going to use the same strategy as in Section 7.2, proving first that problem (7.58)–(7.59) has a solution $u_{c,R}$ (Lemma 7.22). The proof that the constraint is qualified cannot be made as in the 2D case. Then, we show that $u_{c,R}$ satisfies an equation similar to (7.14), derive bounds on $u_{c,R}$ (Lemma 7.24 and 7.25), and check that the constraint is not active thanks to an estimate of the momentum by the energy (Lemma 7.26). Then we pass to the limit as R tends to infinity, which yields a solution of (7.14) with boundary conditions (7.3).

The extra difficulty compared to the 2D case comes from the fact that u_0 vanishes on $z = 0$, so that the estimate of the momentum is more involved.

Existence of a solution to $\mathcal{I}_R$

In this subsection, we prove that problem (7.58)–(7.59) has a solution:

Lemma 7.22. *There exist c_0, $R_0 > 0$ such that for any $R > R_0$, any $c \in (0, c_0)$, and any $\delta > 0$, problem (7.58)–(7.59) has a solution $u_{c,R}$ that satisfies*

$$\Delta u - 2i\frac{c}{1+\lambda}\partial_x u + (z - |u|^2)u = 0 \quad in \quad \Omega_R, \tag{7.73}$$

for some $\lambda \geq 0$.

Proof: We denote by $\mathcal{X}_R$ the set on which we want to minimize $\mathcal{F}_c$:

$$\mathcal{X}_R = \left\{u \in H^1(\Omega_R), \quad \mathcal{F}_0\left(\frac{u}{u_0}, \Omega_R\right) \leq \delta, \quad u \quad \text{satisfies} \quad (7.59)\right\}.$$

Let us first check that $\mathcal{X}_R$ is not empty. Consider the function $u = u_0\frac{\psi_{c,R}}{\psi_0}$, where $\psi_{c,R}$ is a minimizer of problem (7.25), the 2D functions being supposed to be constant with respect to z. This function u clearly satisfies the boundary conditions (7.59). For fixed z, we have $u_0(r, z) \leq \psi_0(r)$, since u_0 is increasing in z, so that

$$\begin{aligned}
\mathcal{F}_0\left(\frac{u}{u_0}, \Omega_R\right) &= \int_{\Omega_R} \frac{u_0^2}{2}\left|\nabla\left(\frac{\psi_{c,R}}{\psi_0}\right)\right|^2 + \frac{u_0^4}{4}\left(1 - \left|\frac{\psi_{c,R}}{\psi_0}\right|^2\right)^2 \\
&< \int_0^1 dz \int_{\omega_R} \frac{\psi_0^2}{2}\left|\nabla\left(\frac{\psi_{c,R}}{\psi_0}\right)\right|^2 + \frac{\psi_0^4}{4}\left(1 - \left|\frac{\psi_{c,R}}{\psi_0}\right|^2\right)^2 \\
&< \int_0^1 F_0\left(\frac{\psi_{c,R}}{\psi_0}\right) dz \leq \delta.
\end{aligned}$$

Hence, $u \in \mathcal{X}_R$, so that the set is not empty.

Next, one easily proves using the same method as in the 2D case that

$$|\mathcal{L}(w, \Omega_R)| \leq K\left(\sqrt{R}\left(\mathcal{F}_0(w, \omega_R)\right)^{3/4} + R\left(\mathcal{F}_0(w, \omega_R)\right)^{1/2}\right),$$

which implies that $\mathcal{F}_c$ is bounded from below on $\mathcal{X}_R$. Here again, any minimizing sequence is weakly compact in $H^1(\Omega_R)$, so we may pass to the limit in the energy and find a minimizer $u_{c,R}$ of (7.58) with boundary conditions (7.59).

Let us prove that the constraints are qualified, namely that there is no u such that $w = u/u_0$ satisfies $\mathcal{F}_0(w) = \delta$, w is a solution of (7.69), and $u_0 w$ satisfies the boundary conditions (7.59). Thanks to our test function above, the minimizer of $\mathcal{F}_0$ in the set $\mathcal{F}_0 \leq \delta$ is such that $\mathcal{F}_0 < \delta$. Thus, if we prove the uniqueness of solutions of (7.69)–(7.59) in the set $\mathcal{F}_0 \leq \delta$, this will imply that there cannot be a critical point with energy $\mathcal{F}_0 = \delta$.

Let us prove the uniqueness by contradiction, which is a consequence of the uniqueness and nondegeneracy of u_0. Let us assume that there is a sequence c_n tending to 0, and R_n to ∞ such that there are two solutions $u_{1,n}$ and $u_{2,n}$ of (7.69)–(7.59) in Ω_{R_n}, with $\mathcal{F}_0 \leq \delta$. The L^∞ bounds on the solutions and the gradient (see Proposition 7.24 below) allow us to pass to the limit in n and get that $u_{1,n}$ converges to some $\overline{u}$, which is a solution of (7.54)–(7.55) in Ω, with finite $\mathcal{F}_0$. The uniqueness result of Theorem 7.14 implies that in fact $\overline{u} = u_0$. Similarly, $u_{2,n}$ converges to u_0.

Let

$$v_n = \frac{u_{1,n} - u_{2,n}}{\|u_{1,n} - u_{2,n}\|_\infty},$$

which satisfies homogeneous Dirichlet boundary conditions on $\partial\Omega_{R_n}$. By usual elliptic estimates, the maximum of $|v_n|$ cannot be achieved close to the boundary of the domain: it is achieved at some point bounded away from the boundary uniformly with respect to n. Assume that the maximum of v_n stays in a bounded domain. We thus find that v_n converges to a solution of (7.62), which is impossible since the only solution is zero by step 7 of Section 7.3.1 and the limit of v_n is equal to one somewhere. So it means that the maximum of v_n is achieved at a point x_n tending to infinity. We define $w_n = v_n(\cdot + x_n)$. Then w_n converges to a solution of the linearized problem around $p(z)$ (7.68), and we also know that it is zero by Remark 7.17.

Remark 7.23. Here, it is not possible to prove that the constraints are qualified in the same way as in the two-dimensional case. Indeed, the equivalent of Remark 7.6 does not hold because of the boundary condition on $z = 1$: we have uniqueness for equation (7.54), which indeed is the derivative of $\mathcal{F}_0$, but with boundary conditions (7.55), which is a different boundary condition from (7.59) due to the presence of $\psi_{c,R}$. The boundary condition (7.59) implies that the solution is complex-valued; hence the uniqueness proof that we have used in Section 7.3.1 does not work as such.

Bounds on the solutions of $\mathcal{I}_R$

This subsection is the equivalent of Section 7.2.3 for the present three-dimensional case: we prove bounds on the minimizer $\psi_{c,R}$ of (7.58)–(7.59).

Lemma 7.24. *Let $R > 1$, $c > 0$, and $\lambda \geq 0$. Let $u_{c,R}$ be a solution of (7.73) with boundary conditions (7.59). Let $w = \frac{u}{u_0}$. There exists a constant K independent of R, λ, and c such that*

(i) $\|u_{c,R}\|^2_{L^\infty(\Omega_R)} \leq 1 + c^2$,
(ii) $\|\nabla u_{c,R}\|^2_{L^\infty(\Omega_R)} \leq K(1 + c^2)^3$,
(iii) $\|w\|^2_{L^\infty(\Omega_R)} \leq K(1 + c^2)^3$,
(iv) $\|\nabla w\|^2_{L^\infty(\Omega_R)} \leq K(1 + c^2)^4$.

Proof: The proof of (i), (ii), and (iii) follows exactly the same lines as the corresponding ones in Lemma 7.9. Turning to the proof of (iv), we may carry out the same proof to have

$$\|\nabla w\|_{L^\infty(\Omega_R \cap \{z > \frac{1}{2}\})} \leq K(1 + c^2)^3.$$

In order to show that the same inequality holds near $z=0$, we use Taylor expansions of $u_{c,R}$, $\nabla u_{c,R}$, u_0, and ∇u_0, with respect to z, and the equality $\nabla w = \frac{\nabla u_{c,R}}{u_0} - \frac{\nabla u_0 u_{c,R}}{u_0^2}$. The proof follows exactly the same lines as those of (iv) of Lemma 7.9.

We next prove here again that if $\mathcal{F}_0(w)$ is small and if ∇w is suitably bounded, then $|w|$ is close to 1:

Lemma 7.25. *There exist $K>0$ and $\delta_0>0$ depending only on the unique solution u_0 of (7.54)–(7.55) such that, for any $w \in W^{1,\infty}(\Omega_R)$ and any*

$$\delta \le \inf\left\{\delta_0, \frac{K}{\|\nabla w\|_\infty^{12}}\right\},$$

$\mathcal{F}_0(w,\Omega_R) \le \delta$ implies that

$$\frac{1}{2} \le |w| \le \frac{3}{2}. \tag{7.74}$$

Proof: We use exactly the same strategy as in Lemma 7.10, with different powers of δ for α and η. For instance, $\alpha = \delta^{1/8}$ and $\eta = 4\delta^{1/6}$ is a suitable choice. The first case is treated exactly in the same way (here, the inequality $|B(\mathbf{x}_0,\eta)\cap\omega_R| \ge \frac{\pi}{3}\eta^2$ is replaced by $|B(\mathbf{x}_0,\eta)\cap\Omega_R| \ge \frac{1}{6}\frac{4\pi}{3}\eta^3$, valid if R is large enough). The only slight difference is in the second case, where one may need to shift $\mathbf{x}_0$ away from $\{z=0\}$, instead of shifting it away from $B_1\times(0,1)$. If $\mathbf{x}_0$ is close to $\{r=1\}$, and away from $\{z=0\}$, we define $\mathbf{x}_1$ as in the 2D case. If $\mathbf{x}_0$ is close to $\{z=0\}$, and away from $\{r=1\}$, we define $\mathbf{x}_1 = \mathbf{x}_0 + z\alpha/\beta\mathbf{e}_z$, where β comes from step 5 of Section 7.3.1. If $\mathbf{x}_0$ is close to $\{z=0\}$ and $\{r=1\}$, we use step 6 of Section 7.3.1: for any outward direction ν, $\partial^2 u/\partial\nu^2$ has a sign. Hence moving $\mathbf{x}_0$ in the direction of ν increases u_0 and we can find a suitable $\mathbf{x}_1$.

Estimating the momentum

We now prove an estimate of the momentum $\mathcal{L}$ in terms of $\mathcal{F}_0$. The difficulty in the proof is that near $z=0$, u_0 vanishes, this time on a set of infinite measure. We have to treat this region differently from the 2D case.

Lemma 7.26. *Let $R\ge 2$, $c>0$, and let $u\in H^1(\Omega_R)$ satisfying (7.59) be such that $w=\frac{u}{u_0}$ satisfies (7.74). Then there exists a constant K independent of u, R, and c such that*

$$|\mathcal{L}(w,\Omega_R)| \le K\left(\mathcal{F}_0(w,\Omega_R) + \sqrt{\mathcal{F}_0(w,\Omega_R)}\right). \tag{7.75}$$

Proof: We will use that p/u_0 is bounded for r large and $u_0 - p$ is in $L^2(\Omega)$. As in Lemma 7.11, the fact that w satisfies (7.74), together with $u_0\nabla w \in L^2(\Omega\setminus B_2)$, implies that there exist $\rho,\phi\in H^1(\Omega_R)$ such that $\rho\ge\frac{1}{2}$ and

$$w = \rho e^{i\phi}.$$

Using this equality in the definition of $\mathcal{L}$, we obtain

$$\mathcal{L}(w, \omega_R) = \int_{\Omega_R} \frac{i}{2} u_0^2 \left(w \partial_x \overline{w} - \overline{w} \partial_x w\right) = \int_{\Omega_R} u_0^2 \rho^2 \partial_x \phi.$$

Let $\alpha \in (0, 1)$ (which will be made precise below), and consider separately the integrals over $\{r < 1 + \alpha\} \cap \Omega_R$ and over $\{r > 1 + \alpha\} \cap \Omega_R$. The first one is dealt with exactly as in the proof of Lemma 7.11, giving

$$\left| \int_{1<r<1+\alpha} u_0^2 \rho^2 \partial_x \phi \right| \leq K \alpha^{3/2} \sqrt{\mathcal{F}_0(w, \Omega_R)}, \tag{7.76}$$

where K depends only on u_0.

Turning to the integral over $\{1 + \alpha < r < R\}$, we use the same kind of trick:

$$\begin{aligned} \int_{1+\alpha<r<R} u_0^2 \rho^2 \partial_x \phi = \int_{1+\alpha<r<R} u_0^2 (\rho^2 - 1) \partial_x \phi + \int_{1+\alpha<r<R} (u_0^2 - p^2) \partial_x \phi \\ + \int_{1+\alpha<r<R} p^2 \partial_x \phi, \end{aligned} \tag{7.77}$$

where $p = p(z)$ is the unique solution of (7.17). We consider separately the three terms above: for the second term, we use that $u_0 - p \in L^2$ and $\frac{u_0}{p} \in L^\infty$:

$$\begin{aligned} \left| \int_{1+\alpha<r<R} (u_0^2 - p^2) \partial_x \phi \right| &\leq 2 \left(\int_{1+\alpha<r<R} \frac{(u_0^2 - p^2)^2}{u_0^2} \right)^{\frac{1}{2}} \left(\int_{\Omega_R} u_0^2 \rho^2 \partial_x \phi^2 \right)^{\frac{1}{2}} \\ &\leq 2K \sqrt{2\mathcal{F}_0(w, \Omega_R)}, \end{aligned} \tag{7.78}$$

with $K = \left(1 + \left\| \frac{p}{u_0} \right\|_{L^\infty(B_{1+\alpha}^c \times (0,1))}\right) \|u_0 - p\|_{L^2(\Omega)}$. Turning to the third term of (7.77), we integrate by parts with respect to the first two space coordinates and get

$$\begin{aligned} \left| \int_{1+\alpha<r<R} p^2 \partial_x \phi \right| &= \left| \int_{r=1+\alpha} p^2 \phi n_x \right| = \left| \int_{r=1+\alpha} p^2 \left(\phi - \inf_{r=1+\alpha} \phi \right) n_x \right| \\ &\leq \int_0^1 p(z)^2 \left(\int_{r=1+\alpha} |\nabla \phi|^2 \right)^{\frac{1}{2}} dz \\ &\leq K \|p\|_{L^2(0,1)} \left(\int_{r=1+\alpha} \frac{p(z)^2}{u_0^2} u_0^2 \rho^2 |\nabla \phi|^2 \right)^{\frac{1}{2}} \\ &\leq K \|p\|_{L^2(0,1)} \left\| \frac{p}{u_0} \right\|_{L^\infty(B_{1+\alpha}^c \times (0,1))} \sqrt{2\mathcal{F}_0(w, \Omega_R)}, \end{aligned} \tag{7.79}$$

for a suitable choice of $\alpha \in \left(\frac{1}{2}, 1\right)$. Finally, we deal with the first term of (7.77):

$$\left|\int_{1+\alpha<r<R} u_0^2(\rho^2-1)\partial_x\phi\right| \le \left(\int_{1+\alpha<r<R} u_0^2|\nabla\phi|^2\right)^{\frac{1}{2}} \left(\int_{1+\alpha<r<R} u_0^2(\rho^2-1)^2\right)^{\frac{1}{2}}$$
$$\le \sqrt{2\mathcal{F}_0(w,\Omega_R)}\left(\int_{1+\alpha<r<R} u_0^2(\rho^2-1)^2\right)^{\frac{1}{2}}. \quad (7.80)$$

Collecting (7.80), (7.78), (7.79) allows to bound the term considered in (7.77), and yields

$$\left|\int_{1+\alpha<r<R} u_0^2\rho^2\partial_x\phi\right| \le K\sqrt{\mathcal{F}_0(w,\Omega_R)}\left(1+\left(\int_{1+\alpha<r<R} u_0^2(\rho^2-1)^2\right)^{\frac{1}{2}}\right). \quad (7.81)$$

In order to bound the right-hand side of (7.81), we split the integral into an integral over $\{\beta<z<1\}\cap\{1+\alpha<r<R\}$ and an integral over $\{0<z<\beta\}\cap\{1+\alpha<r<R\}$, for some $\beta>\frac{1}{4}$ to be made precise below. The first one is dealt with using the fact that u_0 is bounded below:

$$\int_{\substack{1+\alpha<r<R,\\ \beta<z<1}} u_0^2(\rho^2-1)^2 \le K\int_{\substack{1+\alpha<r<R,\\ \beta<z<1}} u_0^4(\rho^2-1) \le K\mathcal{F}_0(w,\Omega_R). \quad (7.82)$$

The second one is treated as follows:

$$\int_{\substack{1+\alpha<r<R,\\ 0<z<\beta}} u_0^2(\rho^2-1)^2 \le K\int_{\substack{1+\alpha<r<R,\\ 0<z<\beta}} z^2(\rho^2-1)^2. \quad (7.83)$$

We now integrate by parts with respect to z, getting

$$\left|\int_{\substack{1+\alpha<r<R,\\ 0<z<\beta}} z^2(\rho^2-1)^2\right| = \frac{1}{3}\left|\int_{\substack{1+\alpha<r<R,\\ z=\beta}} z^3(\rho^2-1)^2 - \int_{\substack{1+\alpha<r<R,\\ 0<z<\beta}} z^3(\rho^2-1)\rho\partial_z\rho\right|$$
$$\le \frac{K}{\beta}\int_{\substack{1+\alpha<r<R,\\ z=\beta}} u_0^4(\rho^2-1)^2$$
$$+K\left(\int_{\substack{1+\alpha<r<R,\\ 0<z<\beta}} u_0^4(\rho^2-1)^2\right)^{\frac{1}{2}}\left(\int_{\substack{1+\alpha<r<R,\\ 0<z<\beta}} u_0^2|\nabla\rho|^2\right)^{\frac{1}{2}}$$
$$\le \frac{K}{\beta}\int_{\substack{1+\alpha<r<R,\\ z=\beta}} u_0^4(\rho^2-1)^2 + K\mathcal{F}_0(w,\Omega_R). \quad (7.84)$$

Here, we point out, as in the proof of Lemma 7.11, that β may chosen greater than $\frac{1}{4}$ and such that $\int_{\substack{1+\alpha<r<R,\\ z=\beta}} u_0^4(\rho^2-1)^2 \le 16\mathcal{F}_0(w,\Omega_R)$. Hence, the right-hand side of (7.83) is bounded by $K\mathcal{F}_0(w,\Omega_R)$. Inserting this estimate in (7.81) gives

$$\left|\int_{1+\alpha<r<R} u_0^2\rho^2\partial_x\phi\right| \le K\left(\sqrt{\mathcal{F}_0(w,\Omega_R)}+\mathcal{F}_0(w,\Omega_R)\right).$$

This, together with (7.76), concludes the proof of (7.75).

The result above allows us to prove the following:

Proposition 7.27. *There exist* $\delta_1 > 0$ *and* $K > 0$ *such that for all* $R \geq 2$*, all* $\delta \leq \delta_1$*, and all* $c \in (0, K\sqrt{\delta})$*, any solution* $u_{c,R}$ *of (7.58) with boundary conditions (7.59) satisfies* $\mathcal{F}_0\left(\frac{u_{c,R}}{u_0}, \Omega_R\right) < \delta$*. In addition,* $u_{c,R}$ *is a solution of*

$$\Delta u - 2ic\partial_x u + (1 - |u|^2)u = 0 \quad in \quad \Omega_R. \tag{7.85}$$

Proof: Let $c < c_0$ (where c_0 is defined in Lemma 7.22), $\delta \leq 1$, and let $u_{c,R}$ be a solution of (7.58). Applying Lemma 7.22 and Lemma 7.24, we find that there is a constant $K_1 > 0$ independent of R, c, and δ such that if $w = \frac{u_{c,R}}{u_0}$, then $\|\nabla w\|_\infty \leq K_1$. Hence, applying Lemma 7.25, we find that there exists some $\delta_1 > 0$ independent of R, c, and δ such that if $\delta \leq \delta_1$, $\mathcal{F}_0(w) \leq \delta$ and w satisfies (7.39).

We want to show that the constraint is not active. We apply Lemma 7.26 and find that for some constant K_2 independent of R, c, and δ, we have $|\mathcal{L}(w)| \leq K_2(\mathcal{F}_0(w) + \sqrt{\mathcal{F}_0(w)}) \leq 2K_2\sqrt{\mathcal{F}_0(w)}$. Assume that the minimum of $\mathcal{F}_c$ is achieved by some w such that $\mathcal{F}_0(w) = \delta$. This implies that

$$\mathcal{F}_c(w) \geq \delta - 2cK_2\sqrt{\delta}.$$

Now, we may also apply Lemma 7.22 with $\frac{\delta}{2}$ instead of δ. We thus obtain $\tilde{w} = \frac{\tilde{u}_{c,R}}{u_0}$ such that $\tilde{u}_{c,R}$ is a solution of problem (7.58), $\mathcal{F}_0(\tilde{w}) \leq \frac{\delta}{2}$, and all the estimates above are valid with $\frac{\delta}{2}$ instead of δ. This implies that

$$\mathcal{F}_c(\tilde{w}) \leq \frac{\delta}{2} + cK_3\sqrt{2\delta}.$$

But $\tilde{w}$ is also a test function for problem (7.58); hence $\mathcal{F}_c(\tilde{w}) \geq \mathcal{F}_c(w)$, which implies

$$\frac{\delta}{2} + cK_3\sqrt{2\delta} \geq \delta - 2cK_2\sqrt{\delta}, \quad \text{and hence} \quad cK_4 \geq \sqrt{\delta}.$$

If $c < \frac{\sqrt{\delta}}{K_4}$, we reach a contradiction. This implies that for the minimizer, $\mathcal{F}_0(w) < \delta$, so that the constraint in (7.58) is not active, and the Lagrange multiplier must be zero: $u_{c,R}$ satisfies (7.85).

End of the Proof of Theorem 7.16

We now conclude the proof of Theorem 7.16. We apply Proposition 7.12, and find that for some $c_0 = K\sqrt{\delta_1}$, there exists a solution $u_{c,R}$ of (7.85) with boundary conditions (7.59). In addition, this function satisfies

$$F_0\left(\frac{u_{c,R}}{u_0}\right) \leq \delta, \quad \|u_{c,R}\|_{W^{1,\infty}(\Omega_R)} \leq K, \tag{7.86}$$

for some constant K independent of R. We thus can extract weak convergence in H^1_{loc} and strong convergence in L^4_{loc}, allowing us to pass to the limit in the energy bound above and in the equation.

The fact that this solution u_c is vortex-free comes from the fact $\frac{1}{2} \le \left|\frac{u_c}{u_0}\right| \le \frac{3}{2}$, and it also has finite energy, a property inherited from $u_{c,R}$.

The convergence part is proved exactly in the same way as in the proof of Theorem 7.4, using the uniqueness of u_0 to obtain convergence in $L^\infty_{\mathrm{loc}}(\Omega)$, and Lemma 7.28 below to deal with infinity.

There remains only to prove the uniqueness part of Theorem 7.16, namely, that $\forall M$, $\exists c_0$ such that for $c \le c_0$, there is a unique solution u_c of (7.14)-(7.15) with $\mathcal{F}_0(u_c/u_0) \le M$. The proof uses the nondegeneracy of u_0 in the same spirit as the proof of Lemma 7.22. It goes by contradiction, assuming that there are two such sequences as c tends to 0. We prove that they both tend to u_0 using the uniqueness result of Theorem 7.14 and that their renormalized difference tends to a solution of the linearized problem at u_0 or $p(z)$, which contradicts the nondegeneracy of u_0 and $p(z)$.

Limit at infinity

We prove here the analogue of Lemma 7.13:

Lemma 7.28. *Let $M > 0$, and let u_c be a solution of (7.14)–(7.15) such that $\mathcal{F}_0(\frac{u_c}{u_0}) \le M$. Assume in addition that the boundary data ψ_c in (7.15) converges to* 1 *at infinity. Then,*

$$\lim_{r\to\infty} u_c(\mathbf{x}) = p(z), \tag{7.87}$$

where p is the solution of (7.17). Moreover, this limit is uniform with respect to $c \to 0$.

Note that according to Lemma 7.13, it is always possible to impose the condition above on ψ_c.

Proof: We use again the notation $w = \frac{u_c}{u_0}$. Hence, we know that (7.87) is true on $\{z = 1\}$. Pointing out that the proof of Lemma 7.24 applies to the present case (indeed, we use in this proof only the fact that $u_{c,R}$ is a solution of (7.73)), we know that $\nabla w \in L^\infty(\Omega)$. Hence, using standard elliptic estimates, we infer that $D^2 w \in L^\infty(\Omega \cap \{z > \alpha\})$, for any $\alpha > 0$. This, together with the fact that $u_0 \nabla w \in L^2(\Omega)$, clearly implies that the function

$$f(x, y) = \int_\alpha^1 u_0^2 |\nabla w|^2(\mathbf{x}) dz$$

converges to zero as (x, y) goes to infinity. Now, $|w(x, y, z) - \psi_c(x, y)| \le \frac{\sqrt{1-z}}{z}\left(\int_z^1 u_0^2 |\nabla w|^2\right)^{1/2}$, so that w converges to 1 at infinity, uniformly on

$\Omega \cap \{z > \alpha\}$, for any $\alpha > 0$. We then note that u_0 converges to $p(z)$ at infinity and that $|u_c| + |u_0| \le Kz$ for some constant K to obtain (7.87).

The fact that this limit is uniform with respect to $c \to 0$ is proved by contradiction: assuming that the limit is not uniform, we have a sequence $c_n \to 0$ and a sequence $\mathbf{x}_n \to \infty$ such that $|u_{c_n}(\mathbf{x}_n) - p(z_n)| > \varepsilon$, for some $\varepsilon > 0$. Considering the sequence $\tilde{u}_n(\mathbf{x}) = u_{c_n}(\mathbf{x} + \mathbf{x}_n)$, we see that it is bounded in $W^{1,\infty}$ and thus converges in L^∞_{loc} to some function u that satisfies (7.64)–(7.65). In addition, $\mathcal{F}_p(u/p) \le \delta$. Applying Remark 7.21, we find that $u = p$, which is a contradiction.

8

Further Open Problems

Many open problems have been described in the course of the book, but we present some extra ones in this chapter, corresponding to new directions.

8.1 Setting in the whole space for the Thomas–Fermi regime

The analysis described in Chapters 3, 4, 6 sets the problem in the bounded domain $\mathcal{D} = \{\rho_{\mathrm{TF}} > 0\}$. But the original problem consists in minimizing the energy in the whole space:

$$E_\varepsilon(u) = \int_{\mathbf{R}^n} \frac{1}{2}|\nabla u|^2 - \mathbf{\Omega} \times \mathbf{r} \cdot (iu, \nabla u) + \frac{1}{4\varepsilon^2}\left((|u|^2 - \rho_{\mathrm{TF}}(\mathbf{r}))^2 - (\rho_{\mathrm{TF}}(\mathbf{r})_-)^2\right) \tag{8.1}$$

under $\int_{\mathbf{R}^n} |u|^2 = 1$, when n =2 or 3 and $(\rho_{\mathrm{TF}})_-$ is the negative part of ρ_{TF}.

8.1.1 Three-dimensional problem

For $n = 3$, there are no results about of the properties of the minimizers of E_ε in $\mathbf{R}^3$.

8.1.2 Two-dimensional problem

For $n = 2$, the problem has been addressed by Ignat–Millot [80, 81] for $\rho_{\mathrm{TF}} = \rho_0 - x^2 - \alpha^2 y^2$. They prove that in $\mathbf{R}^2 \setminus \mathcal{D}$, $|u|$ decays exponentially fast and the properties that they obtain on vortices are only in $\mathcal{D}$. Indeed, energetically favorable vortices are located close to the origin. Information in $\mathbf{R}^2 \setminus \mathcal{D}$ is missing: we do not know whether $|u|$ is small and does not vanish or whether u has vortices in this small density region.

Open Problem 8.1 *There exists a constant C such that for $\Omega < C$, minimizers of E_ε do not have vortices in $\mathbf{R}^2$.*

This would require other tools than energy estimates.

The issue is to understand whether vortices start to exist in $\mathbf{R}^2 \setminus \mathcal{D}$ and for which value of the velocity. Do they appear on a circle in this low-density region?

8.1.3 Painlevé boundary layer

A first step towards the analysis in the low-density region would be to obtain more information on the density profile η_ε, which is the minimizer of E_ε for $\Omega = 0$. We know that η_ε^2 is close to $(\rho_{\mathrm{TF}})_+$. A lower bound for η_ε in $\mathbf{R}^2 \setminus \mathcal{D}$ would be helpful. This would require us to prove that close to $\partial\mathcal{D}$, $\eta_\varepsilon(\mathbf{x})$ behaves like $\varepsilon^{1/3} p(d(\mathbf{x}, \partial\mathcal{D})/\varepsilon^{2/3})$, where p is the solution of

$$p'' + (2s\sqrt{\rho_0} - p^2)p = 0, \;\; p(s) \underset{s\to-\infty}{\rightarrow} 0, \;\; p(s) \underset{s\to\infty}{\sim} \sqrt{2\sqrt{\rho_0}s}. \tag{8.2}$$

One may hope to use sub- and supersolution arguments.

8.1.4 Vortices in the hole

If $\mathcal{D} = \{\rho_{\mathrm{TF}} > 0\}$ is an annulus, then the understanding of vortices in the low density region also concerns the inner ball B. The study of the minimization of E_ε in $\mathcal{D}$ leads to the existence of a circulation for $u/|u|$ on ∂B. A natural question is whether the vortices exist in B under the shape of a giant vortex (single point where u vanishes) or whether u vanishes at isolated zeroes close to the origin in this region of low density. Numerical simulations do not help.

8.2 Other scalings

Other regimes than the small-ε limit are of interest for two-dimensional condensates. When ε is small, the vortex cores are small, while when ε is large, the vortex cores get large and multiple-degree vortices can get stabilized. We go back to the initial scaling of the energy (1.2) and define

$$E_G(u) = \int_{\mathbf{R}^2} \frac{1}{2}|\nabla u|^2 - \Omega \times \mathbf{x} \cdot (iu, \nabla u) + \frac{1}{2}V(x,y)|u|^2 + G|u|^4 \tag{8.3}$$

under $\int_{\mathbf{R}^2} |u|^2 = 1$, where G is our parameter and $V(x,y) = x^2 + y^2$, for instance. When $G = 0$, the analysis in Chapter 5 leads to a very precise description of the critical points of E_0: the eigenstates can be labelled by a radial number n (number of nodes) and a quantum number q (degree). The eigenenergies are $(1-\Omega)q + (1+n)$. We shall concentrate on the states with no radial nodes because they have the lowest energy for a given value of the angular momentum q. When Ω is less than 1, the ground state is for $q = 0$, while for $\Omega = 1$, all the states are degenerate. When G is small, the interaction energy can be considered as a perturbation. The degeneracy at $\Omega = 1$ is lifted: the critical frequencies Ω_q for which the minimizer has a total degree-q split. It turns out that wave functions that are superpositions of states with different q_i generate an array and have less interaction energy than the pure multiply quantized eigenstate u_q with $q = \sum_i q_i$.

Nevertheless, for potentials stronger than harmonic, that is, $V(x, y) = x^s + y^s$ with $s > 2$ or $V(r) = r^2 + kr^4$, the situation is different. In the noninteracting case $G = 0$, pure multiply quantized vortex states have a smaller energy than any superposition of eigenstates having the same total angular momentum or degree. This persists for finite and small G. Numerical simulations have been performed in [82, 105] and bifurcation diagrams are analyzed. No mathematical results have been proved in this regime, where the use of perturbation theory could be helpful.

8.3 Other models

Experiments are moving fast and there are many possible directions of work. We highlight a few of them here with appropriate references.

8.3.1 Optical lattices

There are experiments in which several condensates are confined in an array generated by an optical potential created by a standing laser wave. In this framework, the trapping potential $V(\mathbf{x})$ in (1.1) has to be modified to include the magnetic harmonic potential as before and the optical potential

$$V(\mathbf{x}) = V_h + V_o = (x^2 + \alpha^2 y^2 + \beta^2 z^2) + v_0 \sin^2\left(\frac{2\pi}{\lambda} z\right).$$

The optical potential produces a one-dimensional array of wells separated by $\lambda/2$. The quadratic expansion of the optical potential around the local minima gives rise to an additional harmonic potential with frequency $\sqrt{v_0}2\pi/\lambda$. If this is much larger than β, the magnetic harmonic trapping in the z direction can be neglected. Depending on the values of λ and v_0, equivalent of tunnelling phenomena can be achieved giving rise to collective effects. This is a first experimental step towards Josephson effects, which have just been observed [18]. We refer to [95, 120].

8.3.2 Multicomponent condensates

One may consider a coupled model between multicomponent or spinor Bose–Einstein condensates. It leads to two coupled Gross–Pitaevskii equations for which many patterns have been observed. We refer to [31, 88] for details.

8.3.3 Condensate and noncondensed gas

The Gross–Pitaevskii energy is derived to model the condensate at zero temperature. At a higher temperature, a nonnegligible fraction of the atoms is no longer is the lowest energy state but is excited. They make up what is called the normal gas or thermal gas. The description of the condensate is then an open question. Some authors believe that the Gross–Pitaevskii equation still provides a good description of the whole system: condensed and noncondensed gas [30, 102]. Others [153, 156] have proposed a model that involves coupled equations for the condensate and the noncondensed gas that is a coupled system between the Gross–Pitaevskii equation on the one hand and Boltzmann equations on the other hand.

8.3.4 Fermi gases

Two–component Fermi gases have been achieved experimentally and can exhibit superfluid properties. In the superfluid regime, one expects that a mean-field description, as in the case of bosons, is appropriate. One of the major differences relies on the coefficient a, which is fixed by interaction in the case of bosons. In the Fermi gas of interest, there are only two populated spin components, up and down. The interaction between these two is characterized by the s wave interspecies scattering length a. By applying a magnetic field to a gas of ultracold atoms, it is possible to tune the strength and the sign of a [38]. This phenomenon, known as Feshbach resonance, offers the possibility to study the crossover between a molecular Bose–Einstein condensate and a state described by the Bardeen–Cooper–Schrieffer theory [39].

There is an equation for the wave function describing the relative motion of a spin-up atom with respect to the nearest spin-down atom [126]. When $a > 0$, the solution describes a molecular bound state or dimer. The small-a case corresponds to a Bose condensate of molecules. On the other hand, in the case $a < 0$, there are no bound states and one expects the well-known Bardeen–Cooper–Schieffer model for superconductivity to be valid. The case $|a| = \infty$ is called the unitary gas limit. We refer to the recent overview of [121] for references and prospects of experiments.

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