

**Problem Sheet 10**  
**HORIZONS AND PHOTON SURFACES**

1. a) Consider Lorentz boosts:  $x\partial_t + t\partial_x$  with respect to a pseudo-Cartesian coordinate system on Minkowski space. Verify that these fields are Killing and find their Killing horizons.

b) Kerr spacetime is topologically  $\mathbb{R} \times \mathbb{R}^3$  with coordinates  $(t, r, \vartheta, \varphi)$ , where  $t \in \mathbb{R}$ , the radius  $r > 0$  is sufficiently large, the colatitude  $0 \leq \vartheta \leq \pi$  and the longitude  $0 \leq \varphi \leq 2\pi$  (spherical coordinates on  $\mathbb{R}^3$ ). Introduce the notation

$$\begin{aligned} S &:= \sin \vartheta \\ C &:= \cos \vartheta \\ \rho^2 &:= r^2 + a^2 \cos^2 \vartheta \\ \Delta &:= r^2 - 2Mr + a^2 \\ \mathcal{A} &:= (r^2 + a^2)^2 - \Delta a^2 \sin^2 \vartheta. \end{aligned}$$

Then the Kerr metric on  $\mathbb{R} \times \mathbb{R}^3$  is written as

$$-\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\vartheta^2 - \frac{4MraS^2}{\rho^2} dt d\varphi + \frac{\mathcal{A}}{\rho^2} S^2 d\varphi^2,$$

where  $M^2 \geq a^2$ . Kerr spacetime is not static. It is a more general case of a *stationary spacetime*. Prove that the field  $\partial_t$  is Killing and find its Killing horizon. It is called the *ergosphere*.

2. Is the set of points  $\{x \in M \times \mathbb{R} \mid V(x) = 0, t = 0\}$  in a static spacetime  $(M \times \mathbb{R}, -V^2 dt^2 + g)$  necessarily a section of a Killing horizon?

*Hint:* Consider the metric  $\mathbf{g} = -V^2 dt^2 + \frac{1}{U^2} dr^2 + r^2 g_{\mathbb{S}^{n-2}}$ , where  $V(r) = \sqrt{1 - \frac{2m_1}{r}}$ ,  $U(r) = \sqrt{1 - \frac{2m_2}{r}}$  and  $0 < m_1 < m_2$ . Also determine whether this metric extends to  $r = 2m_1$ .

3. Verify that a time-slice of a photon surface in a static spacetime is a quasi-local photon surface.

4. Show that the boundary of a static manifold with boundary, which is also an asymptotically Schwarzschild system, is a quasi-local photon surface.

5. Let  $(M, g, V)$  be a static manifold with boundary which is an asymptotically Schwarzschild system. Let  $P$  be an equipotential photon surface in  $(\mathbb{R} \times M, -V^2 dt^2 + g)$ . Prove that

- a) the intersection  $P \cap M$  is totally umbilic in  $M$ ;
- b) the mean curvature  $H$  of the intersection  $P \cap M$  is constant;
- c) if  $(\mathbb{R} \times M, -V^2 dt^2 + g)$  satisfies the null energy condition, then in addition to (b)  $H > 0$ .

**Hint:** Use the contracted Gauss equation.

6. Prove that the hypersurface  $r = (mn)^{\frac{1}{n-2}}$  in  $(n+1)$ -dimensional Schwarzschild spacetime is an (equipotential) photon sphere.

7. Consider 4-dimensional Minkowski space. Let  $S$  be the hypersurface given by:

$$-t^2 + dx^2 + dy^2 + dz^2 = a^2, \quad a > 0.$$

Prove that this is an (equipotential) photon surface.

**Hint:** A one-sheeted hyperboloid is a ruled surface. If a line lies on the surface, then it is a geodesic there (why?).