Solution of Gauss-Kuzmin equation

Theorem. *Let $f_0(x);[0,1]\to\mathbb{R}$ be twise continuously differentiable function and*

$$M_0=\max_{x\in[0,1]}|((x+1)f_0'(x))'|=\max_{x\in[0,1]}|(x+1)f_0''(x)+f_0'(x)|.$$

Suppose that $f_0(0)=0$ und $f_0(1)=1$. Consider functions $f_n(x)$ defined by

$$f_{n+1}(x)=\sum_{k=1}^{\infty}\left(f_n\left(\frac{1}{k}\right)-f_n\left(\frac{1}{x+k}\right)\right) \tag{1}$$

then

$$f_n(x)=\frac{\log(x+1)}{\log 2}+\Theta_n(x)q^n \quad mit \quad \max_{n\in\mathbb{Z}_+}\max_{x\in[0,1]}|\Theta_n(x)|\leqslant 3M_0,\quad q=2\zeta(3)-\zeta(2)=0.759^+. \tag{2}$$

Remark. The series in the right hand side of (1) converges because of

$$\left|f_n\left(\frac{1}{k}\right)-f_n\left(\frac{1}{x+k}\right)\right|\leqslant\left|\frac{1}{k}-\frac{1}{x+k}\right|\cdot\max_{x\in[0,1]}|f_n'(x)|\leqslant\frac{\max_{x\in[0,1]}|f_n'(x)|}{k^2}.$$

Substituting in (2) the values $x=0$ and $x=1$ we get

$$f_n(0)=0,\quad f_n(1)=1\quad\forall\, n. \tag{3}$$

Lemma. *Consider function $f(x):[0,1]\to\mathbb{R}$. Suppose that $f(x)\uparrow$ on $[0,1]$. Consider operator*

$$Uf(x)=\sum_{k=1}^{\infty}P_k(x)f\left(\frac{1}{x+k}\right),\quad where\quad P_k(x)=\frac{x+1}{(x+k)(x+k+1)}.$$

Then $Uf(x)\downarrow$ on $[0,1]$.

Proof. We observe that

$$\sum_{k=1}^{\infty}P_k(x)=1\quad\text{for all }\ x, \tag{4}$$

because of

$$\sum_{k=1}^{\infty}P_k(x)=(x+1)\sum_{k=1}^{\infty}\frac{1}{(x+k)(x+k+1)}=(x+1)\sum_{k=1}^{\infty}\left(\frac{1}{x+k}-\frac{1}{x+k+1}\right)=1.$$

Moreover, it is clear that

$$P_1(x)=\frac{1}{x+2}\downarrow.$$

We do not claim monotonicity of $P_2(x)$. We calculate the derivative

$$P_k'(x)=\frac{x^2+(2k+1)x+k(k+1)-(x+1)(2x+2k+1)}{(x+k)^2(x+k+1)^2}=\frac{-(x+1)^2+k^2-k}{(x+k)^2(x+k+1)^2}.$$

So, as $0\leqslant x\leqslant 1$,

$$\text{for }\ k\geqslant 3\ \text{ we have }\ P_k'(x)\geqslant 0\ \text{ and }\ P_k(x)\uparrow. \tag{5}$$

Let $0 \leqslant x \leqslant y \leqslant 1$ and so $f(x) \leqslant f(y)$. We write

$$Uf(y) - Uf(x) = S_1 + S_2,$$

where

$$S_1 = \sum_{k=1}^{\infty} P_k(y) \left(f\left(\frac{1}{y+k}\right) - f\left(\frac{1}{x+k}\right)\right), \quad S_2 = \sum_{k=1}^{\infty} (P_k(y) - P_k(x)) f\left(\frac{1}{x+k}\right).$$

it is clear that $P_k(y) > 0$. As $f(x) \uparrow$ and $\frac{1}{x+k} \geqslant \frac{1}{y+k}$ we have $f\left(\frac{1}{x+k}\right) \geqslant f\left(\frac{1}{y+k}\right)$, and so $S_1 \leqslant 0$.

Now we prove that $S_2 \leqslant 0$. We use (4) to see that

$$\sum_{k=1}^{\infty} (P_k(y) - P_k(x)) = 0 \quad \forall\, x, y.$$

So

$$S_2 = \sum_{k=1}^{\infty} (P_k(y) - P_k(x)) f\left(\frac{1}{x+k}\right) - \underbrace{\sum_{k=1}^{\infty} (P_k(y) - P_k(x))}_{=0} f\left(\frac{1}{x+1}\right) =$$

$$= -\sum_{k=1}^{\infty} \left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+k}\right)\right)(P_k(y) - P_k(x)) = -\sum_{k=2}^{\infty} \left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+k}\right)\right)(P_k(y) - P_k(x)) =$$

$$= -\left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right)\right)(P_2(y) - P_2(x)) - \sum_{k=3}^{\infty} \left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+k}\right)\right)(P_k(y) - P_k(x)).$$

By (5) in the last sum $P_k(y) - P_k(x) \geqslant 0$, while $f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right) \leqslant f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+k}\right)$. So

$$S_2 \leqslant -\sum_{k=2}^{\infty} \left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right)\right)(P_k(y) - P_k(x)) = -\left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right)\right) \sum_{k=2}^{\infty} (P_k(y) - P_k(x)) =$$

$$= -\left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right)\right)(1 - P_1(y) - (1 - P_1(x)) = \left(f\left(\frac{1}{x+1}\right) - f\left(\frac{1}{x+2}\right)\right)(P_1(y) - P_1(x))$$

(we use (4)). But $P_1(y) \leqslant P_1(x)$ by monotonicity of P_1. So, $S_2 \leqslant 0$.□

Proof of Theorem. Instead of (1) we consider equation, obtained by differentiation from (1):

$$f'_{n+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f'_n\left(\frac{1}{x+k}\right).$$

Moreover, we will deal with the functions

$$g_n(x) = (x+1) f'_n(x), \tag{6}$$

which satisfy

$$g_{n+1}(x) = U g_n(x),$$

because of

$$\frac{g_{n+1}(x)}{x+1} = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} \frac{g_n\left(\frac{1}{x+k}\right)}{\frac{1}{x+k} + 1} = \sum_{k=1}^{\infty} \frac{1}{(x+k)(x+k+1)} g_n\left(\frac{1}{x+k}\right).$$

As

$$P_k(x) = \frac{k}{x+k+1} - \frac{k-1}{x+k},$$

we see by differentiation that

$$g'_{n+1}(x) = -\sum_{k=1}^{\infty} \frac{k}{(x+k+1)^2}\left(g_n\left(\frac{1}{x+k}\right) - g_n\left(\frac{1}{x+k+1}\right)\right) - \sum_{k=1}^{\infty} \frac{P_k(x)}{(x+k)^2} g'_n\left(\frac{1}{x+k}\right).$$

By Lagrange Theorem,

$$g_n\left(\frac{1}{x+k}\right) - g_n\left(\frac{1}{x+k+1}\right) = \left(\frac{1}{x+k} - \frac{1}{x+k+1}\right) g'_n\left(\frac{1}{x+\theta_k(x)}\right)$$

for some $\theta_k(x) \in [k.k+1]$. So

$$g'_{n+1}(x) = -\sum_{k=1}^{\infty} \frac{k}{(x+k+1)^2}\left(g_n\left(\frac{1}{x+k}\right) - g_n\left(\frac{1}{x+k+1}\right)\right) - \sum_{k=1}^{\infty} \frac{P_k(x)}{(x+k)^2} g'_n\left(\frac{1}{x+k}\right).$$

Denote

$$M_n = \max_{x\in[0,1]} |g'_n(x)|.$$

Then

$$M_{n+1} \leqslant M_n \max_{x\in[0,1]} \left(\sum_{k=1}^{\infty} \frac{k}{(x+k)(x+k+1)^3} + \sum_{k=1}^{\infty} \frac{P_k(x)}{(x+k)^2}\right). \tag{7}$$

By Lemma applied with $f(x) = x^2$ we see that the function

$$x \mapsto \sum_{k=1}^{\infty} \frac{P_k(x)}{(x+k)^2}$$

is decreasing, and so

$$\sum_{k=1}^{\infty} \frac{P_k(x)}{(x+k)^2} \leqslant \sum_{k=1}^{\infty} \frac{1}{k^3(k+1)}.$$

As for the first element in max in (7), we have

$$\sum_{k=1}^{\infty} \frac{k}{(x+k)(x+k+1)^3} \leqslant \sum_{k=1}^{\infty} \frac{1}{(k+1)^3}.$$

Now (7) turns into

$$M_{n+1} \leqslant M_n \sum_{k=1}^{\infty} \left(\frac{1}{(k+1)^3} + \frac{1}{k^3(k+1)}\right) = M_n(2\zeta(3) - \zeta(2)) = M_n q.$$

So

$$\max_{x\in[0,1]} |g'_n(x)| = M_n \leqslant M_0 q^n.$$

Integration from 0 to $x \in [0,1]$ gives

$$g_n(x) = g_n(0) + \Theta_n^{[1]}(x), \quad \Theta_n^{[1]}(x) = \int_0^x g'_n(t)dt, \quad \max_{x\in[0,1]} |\Theta_n^{[1]}(x)| \leqslant M_0 q^n.$$

So by (6),

$$f_n'(x) = \frac{g_n(0) + \Theta_n^{[1]}(x)}{x+1}.$$

Now

$$f_n(x) = \int_0^x \frac{g_n(0) + \Theta_n^{[1]}(t)}{t+1} dt = g_n(0) \cdot \log(x+1) + \Theta_n^{[2]}(x), \quad \Theta_n^{[2]}(x) = \int_0^x \frac{\Theta_n^{[1]}(t)}{t+1} dt, \quad \max_{x \in [0,1]} |\Theta_n^{[2]}(x)| \leqslant M_0 q^n.$$

Putting here $x = 1$ and using (3) we get

$$1 = f_n(1) = g_n(0) \log 2 + \Theta_n^{[2]}(1),$$

or

$$g_n(0) = \frac{1}{\log 2} + \Theta_n^{[3]}, \quad |\Theta_n^{[3]}| = \frac{|\Theta_n^{[2]}(1)|}{\log 2} \leqslant 2M_0 q^n.$$

Finally

$$f_n(x) = \frac{\log(x+1)}{\log 2} + \Theta_n(x), \quad \text{where} \quad |\Theta_n^{[2]}(x)| \leqslant 3M_0 q^n.$$

und alles ist bewiesen.□