Аспекты «велосипедной математики»

XXII Летняя школа «современная математика»
имени Виталия Арнольда
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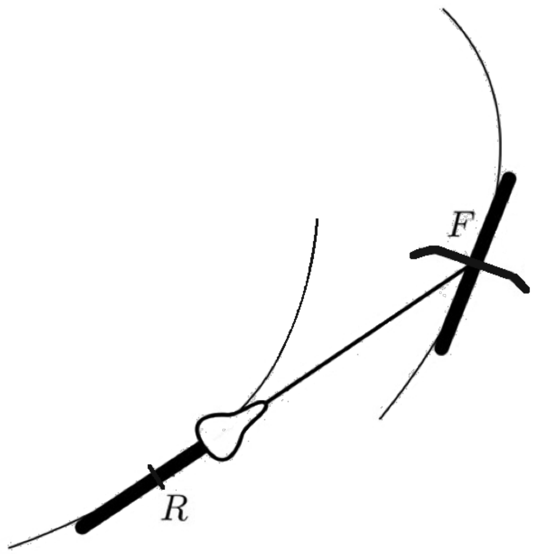
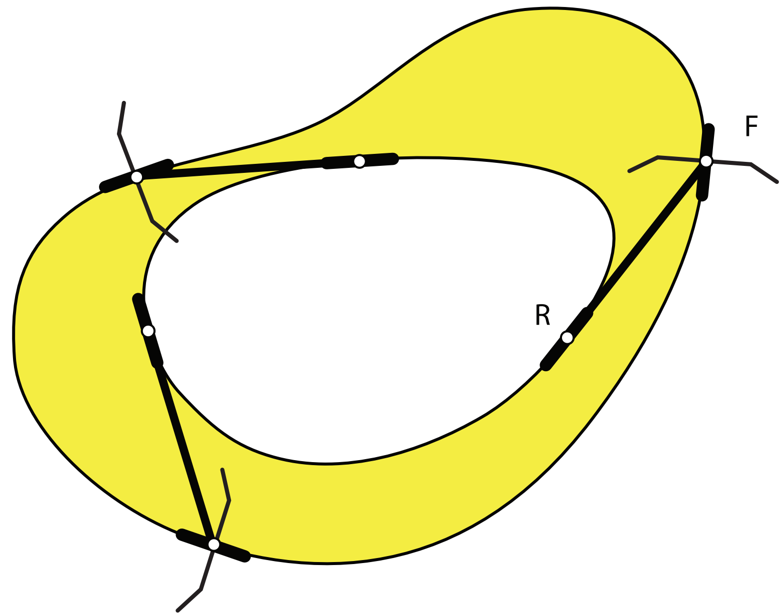
Part 1: The model

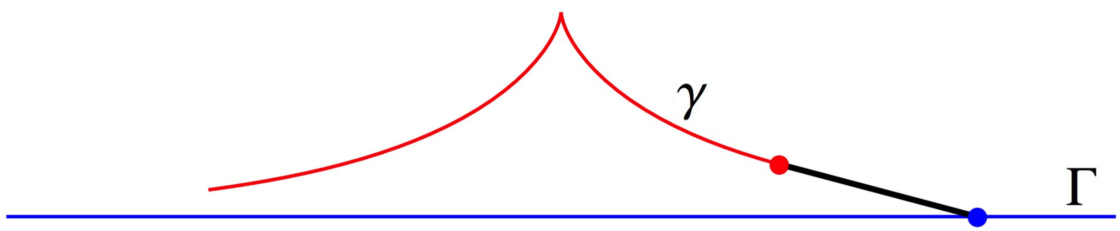




Segment RF moves so that the trajectory of point R is tangent to the segment. The rear track may have cusps, when the steering angle is 90° (not recommended in real life!)

Same in $\mathbf{R}^n$ (and some other Riemannian manifolds).

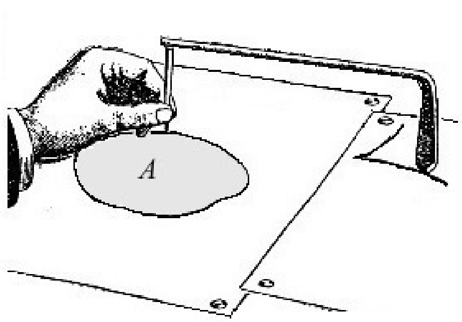
Classical connection: the tractrix is the rear wheel track γ, when the front one, Γ, is a straight line.

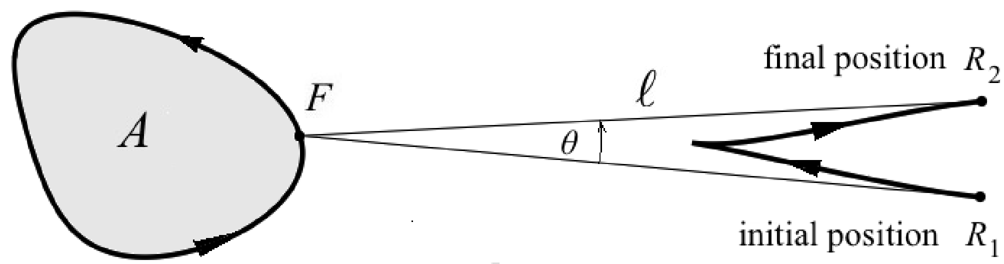


Introduced by Claude Perrault (1613–1688) (a brother of Charles Perrault, of the "Little Red Riding Hood" and "Puss in Boots" fame).

Teaser: *what's the area under the tractrix?*

Hatchet, or Prytz, Planimeter: Holger Prytz, a Danish cavalry officer and engineer, 1886.





How it works:

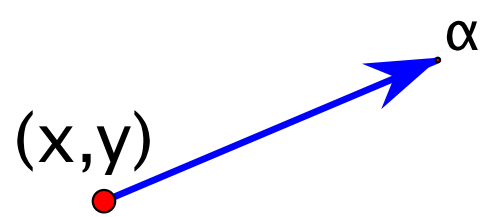
$$\text{Area} = \ell^2\theta + O\left(\frac{1}{\ell}\right),$$

actually, a power series in $1/\ell$.

A consequence for parallel parking: you want to maximize the area bounded by the trajectory of the front wheels.

Contact geometry and sub-Riemannian geometry:

In the plane, the configuration space of a bicycle is the space of contact elements. It has a 2-dimensional distribution, a contact structure, given by the "skating" constraint.



Bicycle motion is a smooth horizontal (Legendrian) curve in this contact space. The projection on the front end is always smooth, and the projection on the rear end may have cusps.

The contact distribution is non-integrable: the bracket of tangent vector fields is not tangent anymore

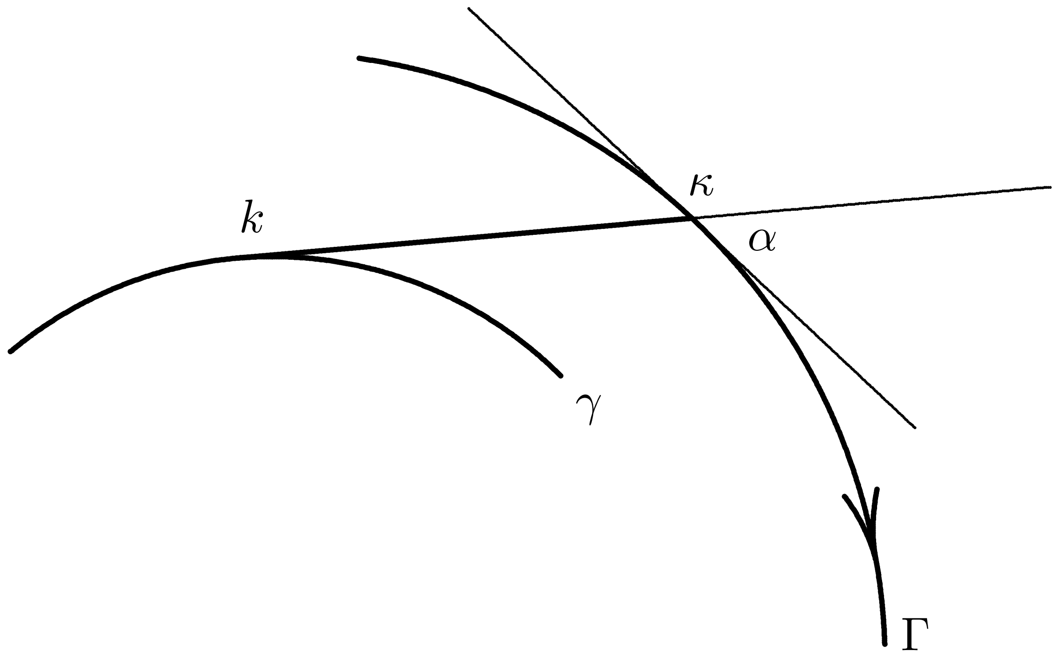
$$\left[\frac{\partial}{\partial\alpha}, \cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y}\right] = -\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}.$$

The Chow-Rashevskii theorem implies that one can connect any two points by a horizontal curve.

Real-life example: parallel parking (again).

Likewise, in $\mathbf{R}^n$, one has a non-integrable n-dimensional distribution in the $(2n-1)$-dimensional configuration space, and the differential of the projection on the front end is a linear isomorphism.

Differential equation determining rear wheel track γ from the front wheel track Γ:



α is the steering angle, x is the arc length parameter on Γ and t on γ, and k and κ are the curvatures of the tracks.

Then

$$\frac{d\alpha(x)}{dx} + \frac{\sin\alpha(x)}{\ell} = \kappa(x), \qquad (*)$$

and also,

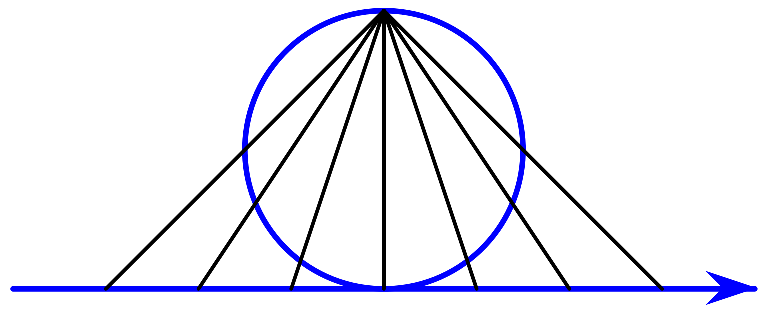
$$\left|\frac{dt}{dx}\right| = |\cos\alpha|, \quad k = \frac{\tan\alpha}{\ell}.$$

Cusps $\equiv$ infinite curvature $\equiv \{\alpha = \pi/2\}$.
Consequence: the front wheel goes faster and should wear out sooner (does it, really?)

This equation also describes the overdamped case of the Josephson effect - you learned about it from Alexei Glutsyuk's talks.

If $y = \tan(\alpha/2)$, then (*) becomes a Riccati equation:

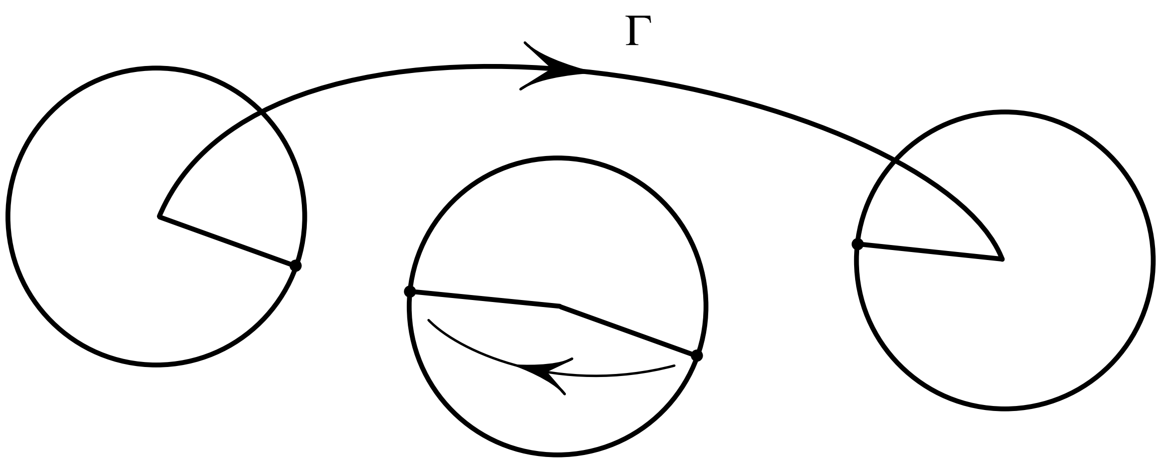
$$y'(x) = -y(x) + (y^2(x) + 1)\kappa(x)/2$$



The change of variables is suggested by the stereographic projection from S^1 to $\mathbf{RP}^1 = \mathbf{R} \cup \infty$.

Part 2: Bicycle monodromy

Monodromy M: initial position $\mapsto$ terminal position (in dimension 2, $M : S^1 \to S^1$, and in dimension 3, $M : S^2 \to S^2$).



Theorem: *For every front track, M is a Möbius transformation, that is, a fractional-linear transformation with real (dimension 2) or complex (dimension 3) coefficients.*

We use that $S^1 = \mathbf{RP}^1$ and $S^2 = \mathbf{CP}^1$, the Riemann sphere.

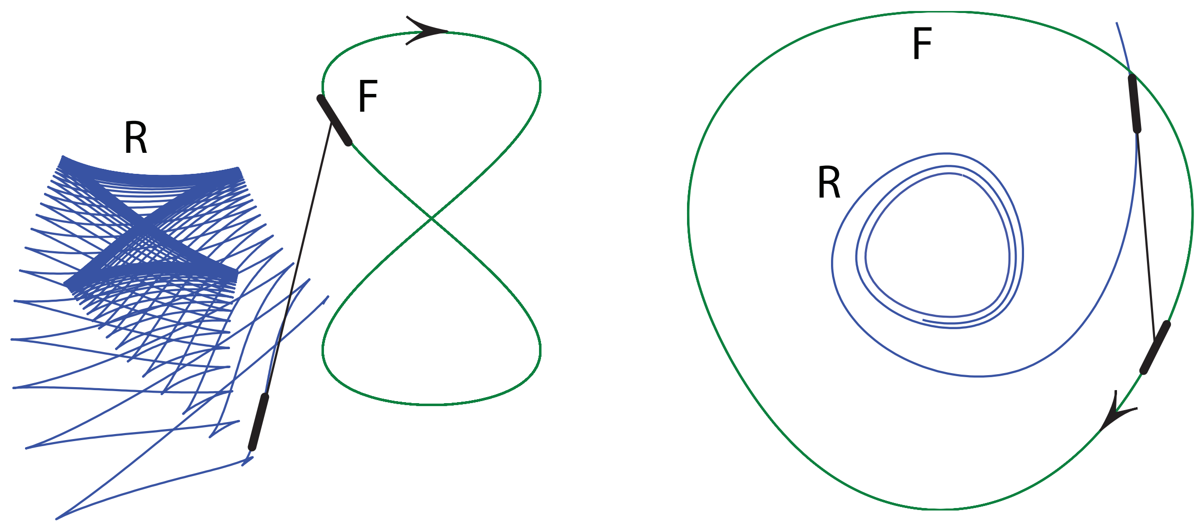
The monodromy is given by the formula

$$M_\Gamma : x \mapsto \frac{ax+b}{cx+d}.$$

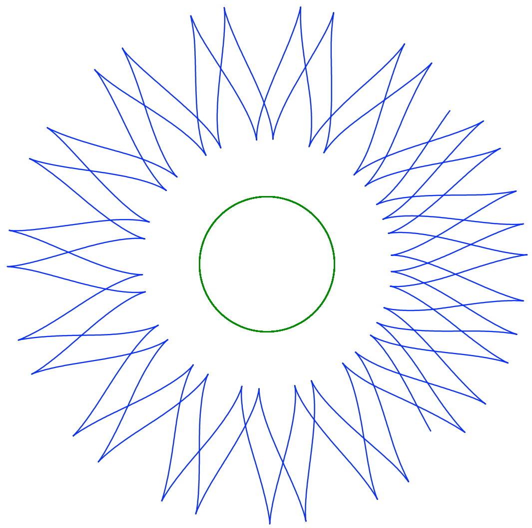
For a closed curve Γ, the bicycle monodromy is well defined up to conjugation.

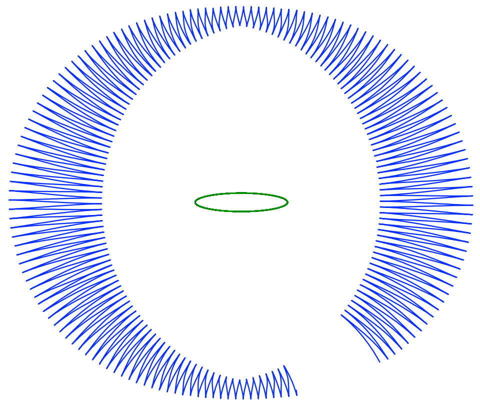
Menzin's conjecture

Hyperbolic monodromy (two fixed points):



Elliptic monodromy (no fixed points):





The case of a circle was studied a long time ago:
F. Morley, *The 'no-rolling" curves of Amsler's planimeter*, Ann. of Math. 1 (1899/00), 21–30.

Menzin's Conjecture (1906): *If* $\ell = 1$ *and* Γ *is an oval of area* $> \pi$*, then the monodromy is hyperbolic.*

> ... the tractrix will approach, asymptotically, a limiting closed curve. From purely empirical observations, it seems that this effect can be obtained so long as the length of arm does not exceed the radius of a circle of area equal to the area of the base curve.

For convex curves, it is now a theorem.

Spherical and hyperbolic versions (S. Howe, M. Pancia, V. Zakharevich):

The monodromy is still Möbius. Differential equations:

$$\frac{d\alpha(x)}{dx} + \cot \ell \ \sin \alpha(x) = \kappa(x), \ \ \frac{d\alpha(x)}{dx} + \coth \ell \ \sin \alpha(x) = \kappa(x).$$

Here $\cot \ell$ and $\coth \ell$ are the curvatures of the circles of radius ℓ in S^2 and H^2.

A curious case, in S^2: if $\ell = \frac{\pi}{2}$ then the the bicycle is parallel translated. If Γ bounds area 2π then the monodromy is the identity: every bicycle path closes up.

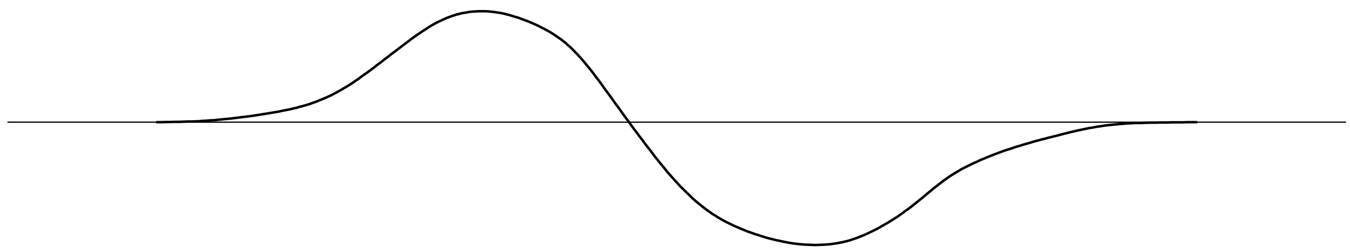
Theorem: *1). in S^2: if Γ is a simple convex curve bounding area $> 2\pi(1 - \cos \ell)$ then the monodromy is hyperbolic;*
2). in H^2: if Γ is a simple horocyclically convex (curvature greater than one) curve bounding area $> 2\pi(\cosh \ell - 1)$ then the monodromy is hyperbolic.

The areas are those of the disks of radius ℓ.

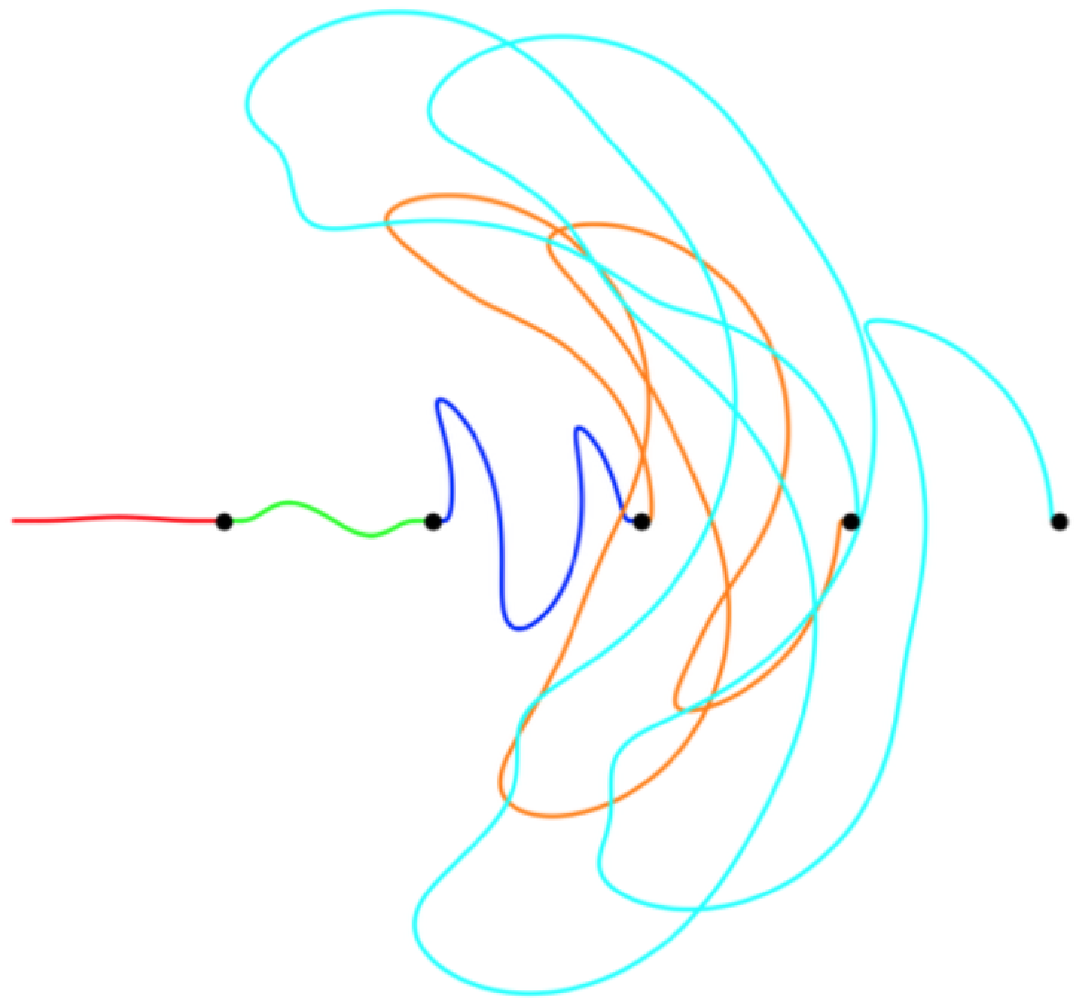
Part 3: Unicycle problem

Can one ride the bike so that it leaves only one track?

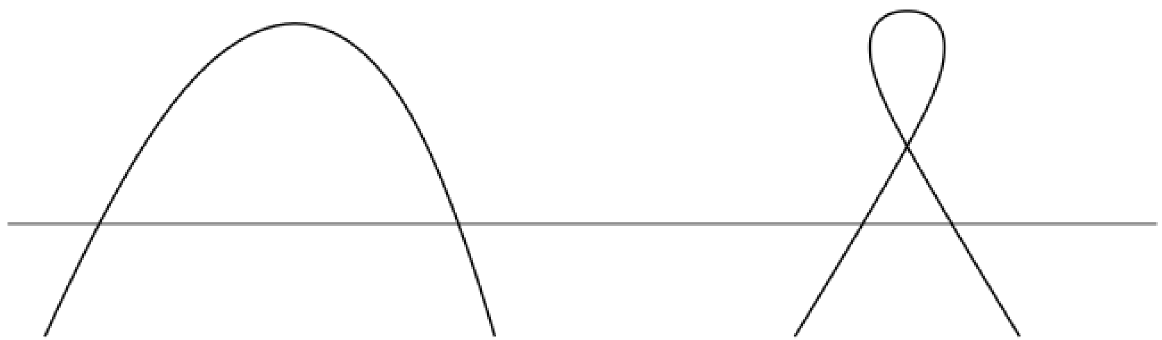
D. Finn's construction: start with a seed curve, tangent to a straight line with all derivatives,



and ride the bike:



Theorem: *The number of 'zeros' on each next segment of the curve is greater than that on the previous one. Likewise for the number of extrema and the number of inflections.*



Proof: Give $\gamma(t)$ the arc length parameterization. Then

$$\Gamma(t) = \gamma(t) + \gamma'(t) = e^{-t}(e^t\gamma(t))'.$$

Each zero of γ is a zero of $e^t\gamma(t)$. By Rolle's Theorem, between two zeros of $e^t\gamma(t)$, there is a zero of its derivative, hence of Γ.

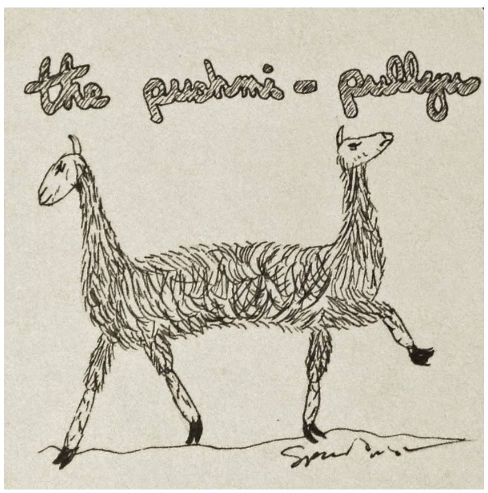
Corollary: *This construction can be extended backward only finitely many times.*

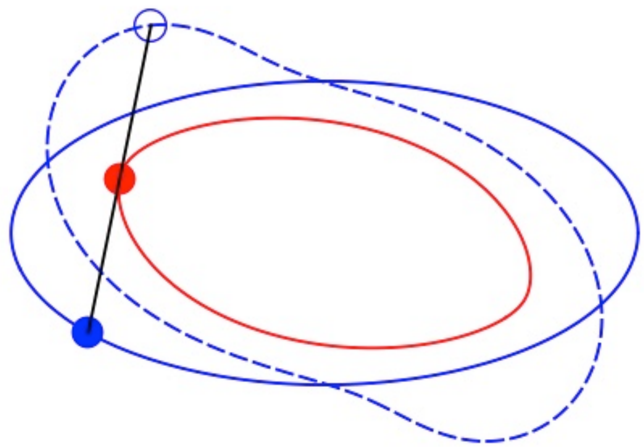
Conjectures: The unicycle track (unless it is straight)
(i) has an infinite amplitude;
(ii) fails to be a graph;
(iii) develops self-intersections;

and any other measure of complexity increases without bound.

Part 4: Bicycle transformation

Two front tracks sharing the rear track:





We say that such front tracks are in the bicycle correspondence: $\mathcal{B}_{2\ell}(\Gamma_1, \Gamma_2)$.

Equivalently, two points, x_1 and x_2, traverse the curves Γ_1 and Γ_2 in such a way that $x_1x_2 = 2\ell$, and the velocity of the midpoint of the segment x_1x_2 is aligned with the segment.

In dimension 2, we get a (generically, 2-2) mapping $\Gamma_1 \mapsto \Gamma_2$, assuming that Γ_1 has a hyperbolic monodromy; in dimension 3, the monodromy always has a fixed point, and no assumptions are needed.

Properties valid in all dimensions

Theorem: *If $\mathcal{B}_{2\ell}(\Gamma_1, \Gamma_2)$ then $M_{\Gamma_1,\lambda}$ and $M_{\Gamma_2,\lambda}$ are conjugated for all values of λ.*

Thus the conjugacy invariants of $M_{\Gamma,\lambda}$, as functions of λ (the spectral parameter), are integrals of the bicycle correspondence. In dimension 2, one may take

$$\frac{\mathrm{Tr}^2}{\det} = \frac{(a+d)^2}{ad-bc}.$$

Theorem [Bianchi permutability]: *Let Γ_1, Γ_2 and Γ_3 be three closed curves, such that $\mathcal{B}_\ell(\Gamma_1, \Gamma_2)$ and $\mathcal{B}_\lambda(\Gamma_1, \Gamma_3)$ hold. Then there exists a closed curve Γ_4, such that $\mathcal{B}_\lambda(\Gamma_2, \Gamma_4)$ and $\mathcal{B}_\ell(\Gamma_3, \Gamma_4)$ hold.*

Informally speaking, the bicycle transformations with different length parameters commute.

Other integrals of the bicycle transformation:

$$\int_{\Gamma} \Gamma(t) \times \Gamma'(t)\ dt$$

(area bivector), and

$$\int_{\Gamma} (\Gamma(t) \cdot \Gamma'(t))\ \Gamma(t)\ dt$$

(centroid). Overall,

$$\binom{n}{2} + n = \binom{n+1}{2},$$

the dimension of the group of motions of $\mathbf{R}^n$ (in agreement with Noether's theorem).

Relation with the filament (binormal, smoke ring, LIE) equation

$$\dot{\Gamma} = \Gamma' \times \Gamma'' = \kappa B.$$

Equation introduced by L. Da Rios (a student of Levi-Civita) in 1906 (the same year as Menzin!). It is a well studied completely integrable systems of soliton type.

Frenet equations:

$$T' = \kappa N, \; N' = -\kappa T + \tau B, \; B' = -\tau N.$$

Theorem: *The bicycle transformation and the flow of the filament equation commute and share the integrals*.

Integrals of the filament equation:

$$\int 1 \; dx, \; \int \tau \; dx, \; \int \kappa^2 \; dx, \; \int \kappa^2 \tau \; dx, \; \int \left((\kappa')^2 + \kappa^2 \tau^2 - \frac{1}{4}\kappa^4 \right) \; dx, \ldots$$

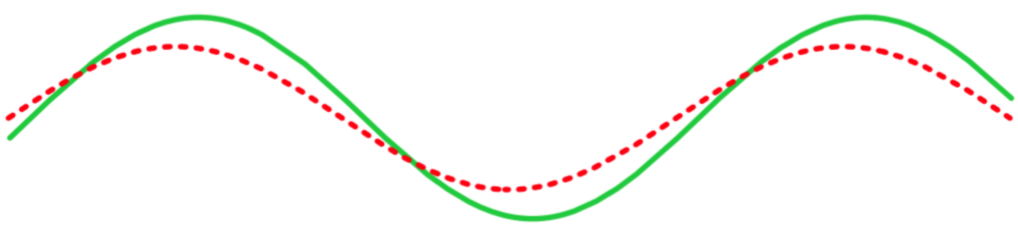
where τ is the torsion and κ is the curvature of a curve.

In dimension 2, $\tau = 0$, and every other integral is non-trivial.

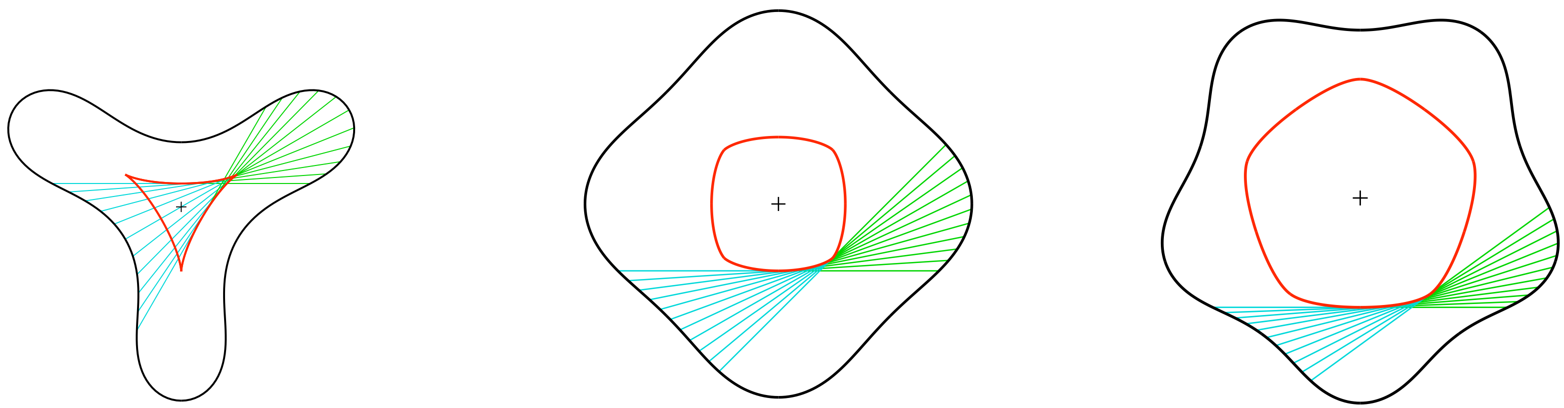
Part 5: Which way did the bicycle go?

In "The Adventure of the Priory School" by A. Conan Doyle, Sherlock Holmes did not do very well:

> No, no, my dear Watson. The more deeply sunk impression is, of course, hind wheel, upon which the weight rests. You perceive several places where it has passed across and obliterated the more shallow mark of the front one. It was undoubtedly heading away from the school.



Usually, you can tell which way the bicycle went, but sometimes you cannot. Trivial example: concentric circles. But also:



and many more: F. Wegner, *Three Problems – One Solution* `http://www.tphys.uni-heidelberg.de/~wegner/Fl2mvs/Movies.html`.

These curves are “self-bicycle".

Ulam's problem: which homogeneous bodies float in equilibrium in all positions? ("Scottish Book", Problem No 19)

In dimension two (floating log), it's the same problem!
(The role of the relative density is played by the relative length of the arc of Γ, subtended by the moving segment.)

Some results:

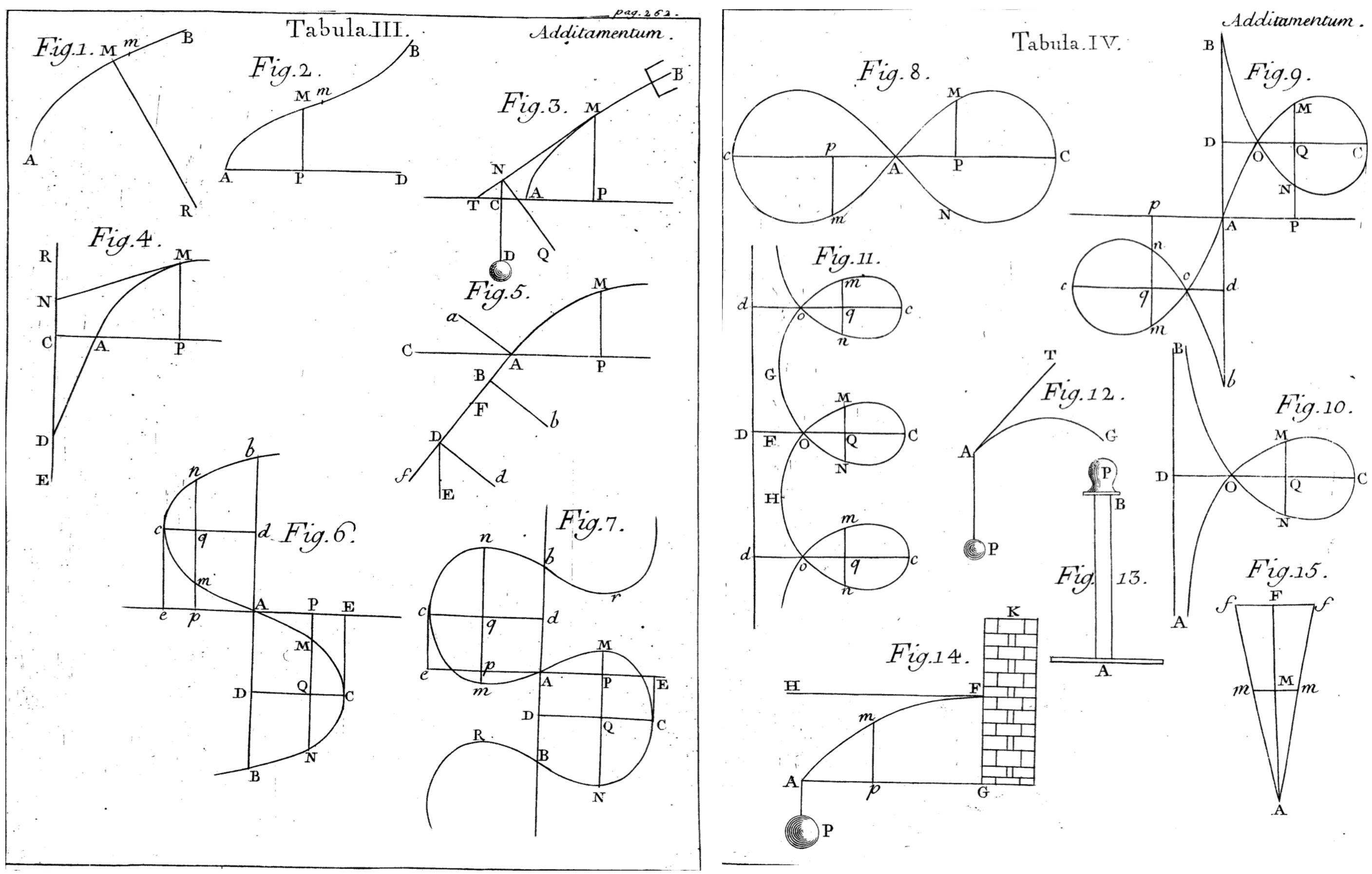
(1) for $\rho = 1/2$, there infinitely many bicycle curves (as many as functions of one variable) (H. Auerbach 1938).
(2) For $\rho = 1/3$ and $\rho = 1/4$, one has rigidity: Γ must be a circle.
(3) *Infinitesimal deformations of a circle.* The unit circle admits a non-trivial infinitesimal deformations iff

$$n \tan(\pi\rho) = \tan(n\pi\rho)$$

for some $n \geq 2$ (“mode locking").

For these values of the parameters, Franz Wegner constructed families of such “bicycle curves".

Classical elastica: extremize the bending energy $\int k^2 dx$ with fixed length, satisfy the Euler-Lagrange equation $k'' + \frac{1}{2}k^3 + \lambda k = 0$.



Buckled ring (or pressurized elastica): relative extrema of the bending energy with perimeter and area constraints:

$$k'' + \frac{1}{2}k^3 + \lambda k + \mu = 0,$$

where λ, μ are Lagrange multipliers.

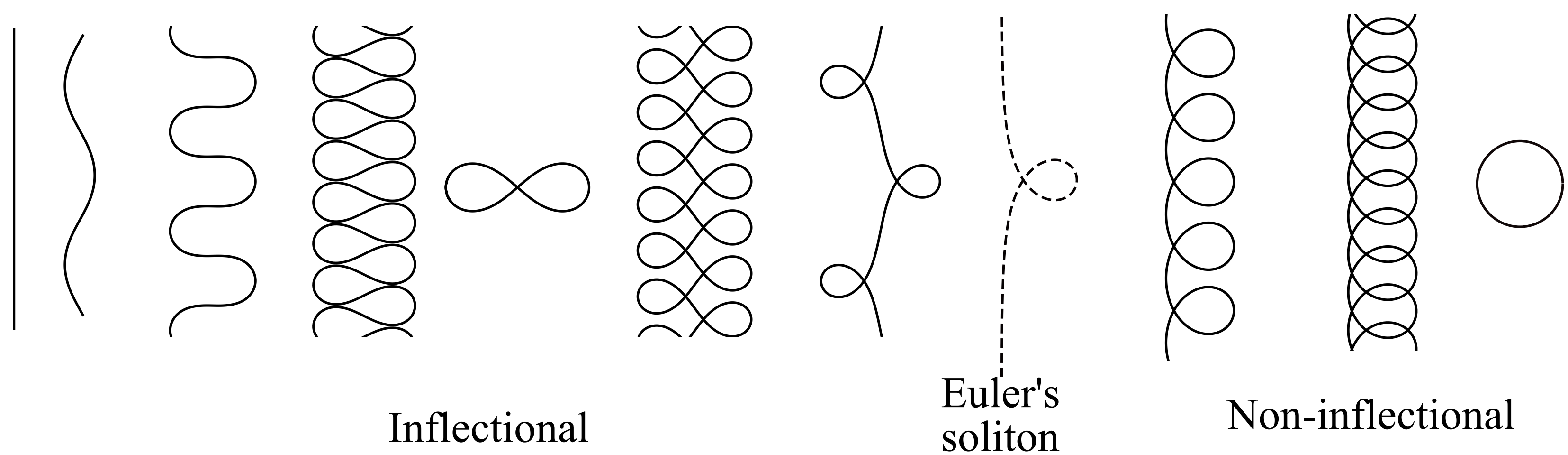
The vector field on curves $X_2 = \frac{k^2}{2}T + k'N$ defines the *planar filament equation*.

Theorem: *The Wegner curves are solitons: under X_2, they evolve by rigid rotations and parameter shifts. They are also buckled rings.*

Part 6: Bicycling geodesics

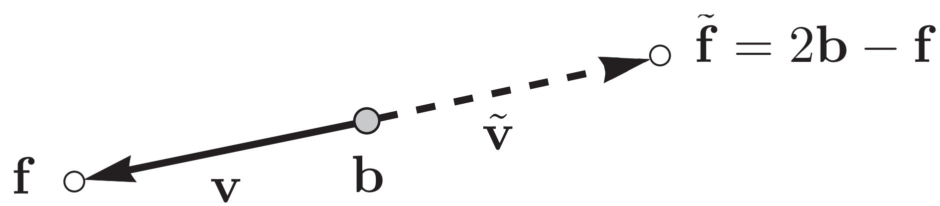
Define the *length of a bicycle path* as the length of the front track, as in sub-Riemannian geometry. What are the *bicycling geodesics*?

A “zoo" of elastic curves:



Remember the tractrix? The Euler soliton is the bicycle transformation of a straight line. This is because of

Lemma: *The flip is a sub-Riemannian isometry.*

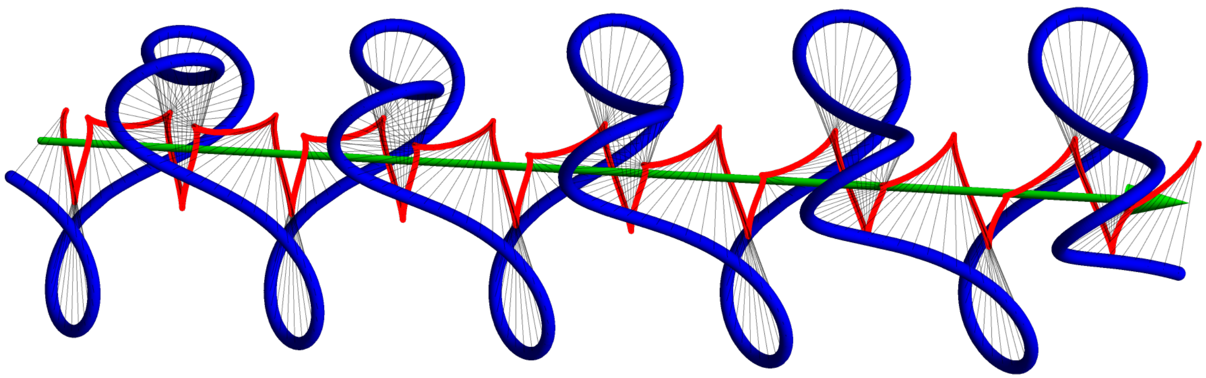


Since a straight line is obviously a geodesic, so is the Euler soliton.

Theorem: *In $\mathbf{R}^2$, the front tracks of the bicycling geodesics are arcs of non-inflectional elastic curves.*

Kirchhoff rods: the extrema of the bending energy among curves with fixed end points, total torsion, and length in $\mathbf{R}^3$.

Theorem: *The front and back tracks of every bicycling geodesic in $\mathbf{R}^n, n \geq 3$, are contained in a 3-dimensional affine subspace, and the front tracks are Kirchhoff rods therein.*

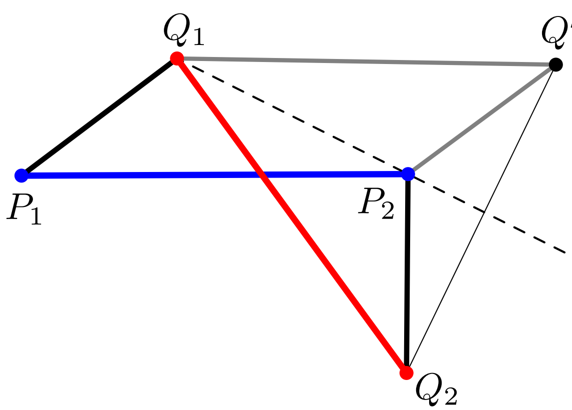


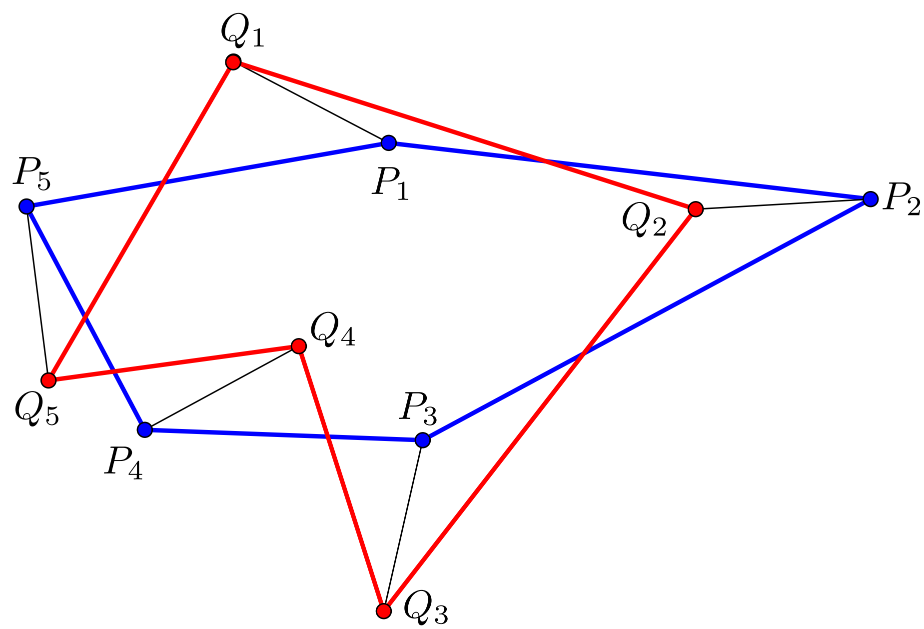
The front track is also the trajectory of a charge in a *Killing magnetic field*.

A magnetic field is a vector field F that exerts the Lorentz force $F \times v$ on the charge moving with the velocity v. A Killing magnetic field is an infinitesimal isometry of $\mathbf{R}^3$, a screw motion, that is, a composition of an infinitesimal rotation about a line and a parallel translation.

Part 7: Discretization

Discrete bicycle monodromy and discrete bicycle correspondence





The main building block is the *Darboux butterfly*.

Theorems:

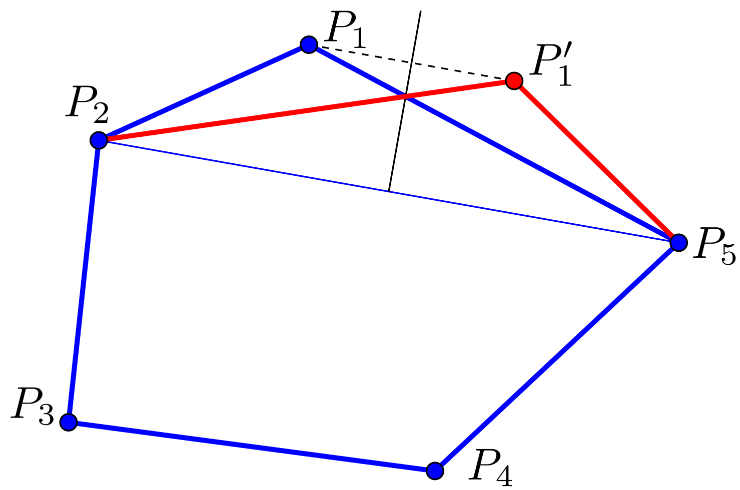
(i) For every polygon P, the monodromy $M_{P,\ell}$ is a Möbius transformation.

(ii) If polygons P_1 and P_2 are in the bicycle correspondence, $\mathcal{B}_\ell(P_1, P_2)$, then $M_{P_1,\lambda}$ and $M_{P_2,\lambda}$ are conjugated for all λ.

(iii) [Bianchi permutability]*: Let P_1, P_2 and P_3 be three polygons, such that $\mathcal{B}_\ell(P_1, P_2)$ and $\mathcal{B}_\lambda(P_1, P_3)$ hold. Then there exists a polygon P_4, such that $\mathcal{B}_\lambda(P_2, P_4)$ and $\mathcal{B}_\ell(P_3, P_4)$ hold.*

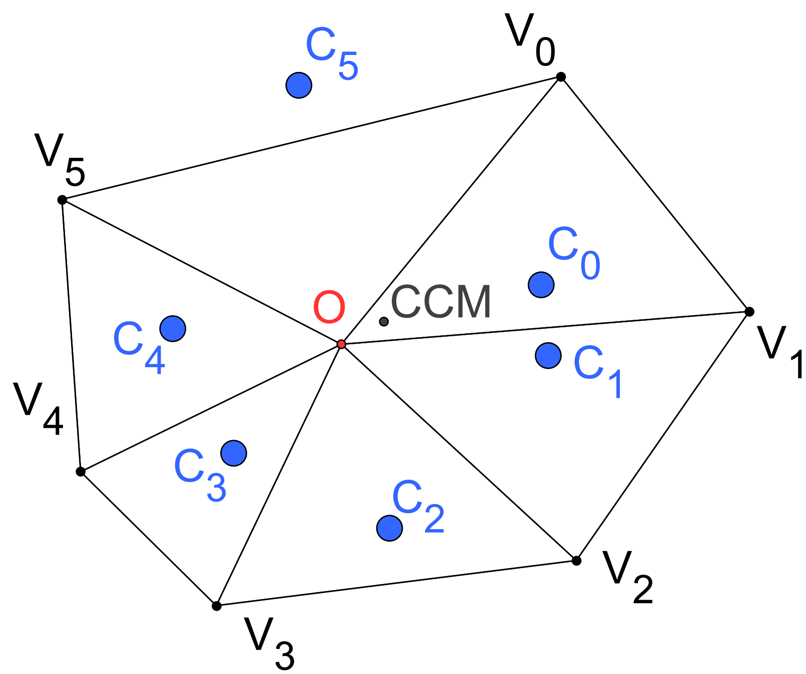
(iv) If P is a Darboux butterfly, then $M_{P,\ell} = Id$ for all λ.

Polygon recutting [Vsevolod Adler]



Theorem: *(i) The bicycle correspondence commutes with the polygon recutting.*
(ii) The conjugacy equivalence class of the monodromy is preserved by the recutting.

Circumcenter of mass: Triangulate a polygon P, take the circumcenter of each triangle with the weight equal to its area, and take the center of mass, $CCM(P)$.



Theorem: $CCM(P)$ *is well defined and it satisfies the Archimedes Lemma: if a polygons is divided into two smaller polygons, then the circumcenter of mass of the compound polygon is the weighted sum of the circumcenters of mass of the two smaller polygons.*

There are versions of this construction in spherical and hyperbolic geometries and in higher dimensions.

Theorem: *The circumcenter of mass is an invariant of the discrete bicycle correspondence and of the polygon recutting in the plane.*

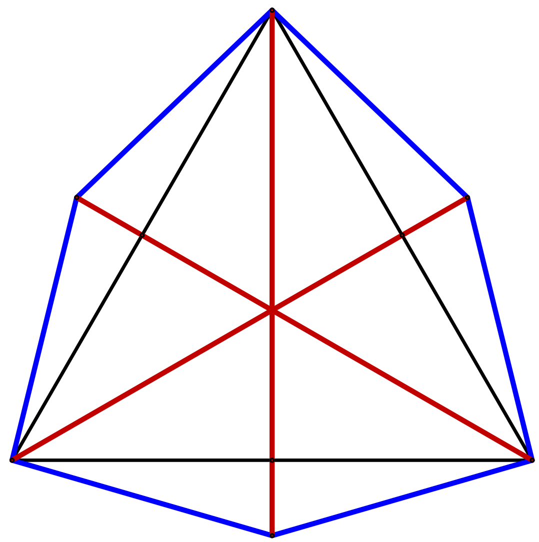
A historical remark (thanks to B. Grünbaum):
C.-A. Laisant, *Théorie et applications des équipollences*. Gauthier-Villars, Paris 1887. On pp. 150–151, the construction is described and attributed to Giusto Bellavitis (1803 –1880).

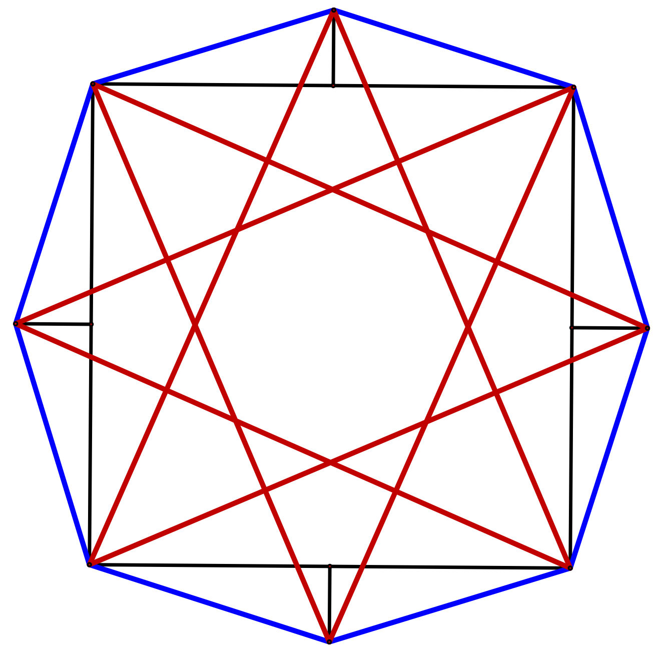
Bicycle (n, k)-gons are convex equilateral n-gons whose k-diagonals have equal length. They are in the discrete bicycle correspondence with themselves.

One has a rigidity problem: *must such a polygon be regular?*

Theorem: *A bicycle (n, k)-gon must be regular in the following cases:*
(i) $k = 2$;
(ii) n *odd and* $k = 3$;
(iii) $n = 2k + 1$;
(iv) $n = 3k$.

But there exist 1-parameter families for odd k and even n:





Theorem: *An infinitesimal deformation of a regular polygon as a bicycle (n, k)-gon exist if and only if*

$$\tan\left(kr\frac{\pi}{n}\right)\tan\left(\frac{\pi}{n}\right) = \tan\left(k\frac{\pi}{n}\right)\tan\left(r\frac{\pi}{n}\right) \qquad (*)$$

for some $2 \leq r \leq n-2$.

In addition to the described solutions ($n = 2r, k$ odd), there may be others.

Theorem [R. Connelly and B. Csikos]: *For $2 \leq r \leq n/2$, all other solutions of equation (*) are given by*

$$k + r = n/2 \quad \text{and} \quad n|(k-1)(r-1),$$

for some r.

Problem: *Do such bicycle polygons really exist?*

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Спасибо!