Lecture 1 part 2: Tammes' problem and irreducible contact graphs

Oleg R. Musin

University of Texas Rio Grande Valley & IITP RAS

Ratmino: July 19, 2019

Papers

O. R. Musin and A. S. Tarasov, *The strong thirteen spheres problem,* Discrete Comp. Geometry, **48** (2012) 128–141.

O. R. Musin and A. S. Tarasov, *Enumeration of irreducible contact graphs on the sphere,* J. of Math Sciences, **203** (2014), 837–850

O. R. Musin and A. S. Tarasov, *Extreme problems of circle packings ...*, Proc. Steklv. Inst.,**288** (2015), 117–131

O. R. Musin and A. S. Tarasov, *The Tammes problem for N=14,* Experimental Math., **24:4** (2015), 460–468

O. R. Musin and A. V. Nikitenko, *Optimal packings of congruent circles on a square flat torus,* Discrete Comp. Geometry, **55:1** (2016), 1–20.

O. R. Musin, Towards a proof of the 24–cell conjecture, *Acta Math Hungar.*, **155** (2018), 184–199

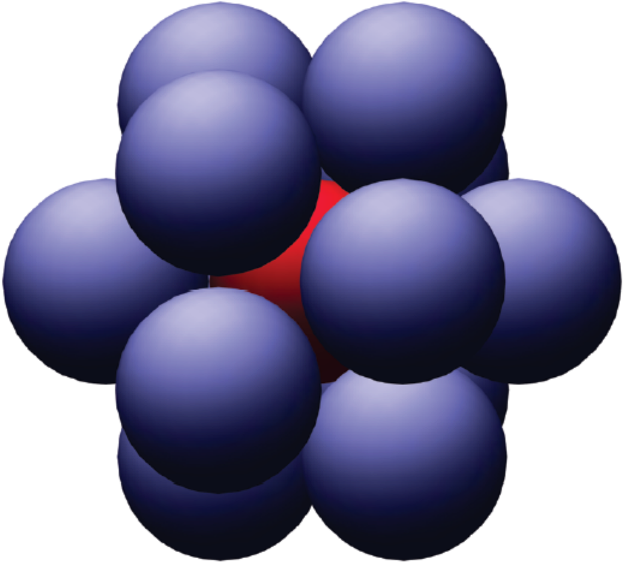
O. R. Musin, *An extension the semidefinite programming bound for spherical codes,* arXiv:1903.05767

Kissing numbers and spherical codes: Methods

I. *Area inequalities.* L. Fejes Tóth (1943); for $d > 3$ Coxeter (1963) and Böröczky (1978)

II. *Distance and irreducible contact graphs.* Schütte, and van der Waerden (1951); Danzer (1963); Leech (1956);...

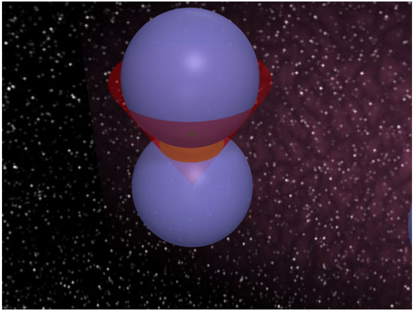
III. *LP and SDP.* Delsarte et al (1977); Kabatiansky and Levenshtein (1978); Odlyzko & Sloane (1978); Bachoc and Vallentin (2008); ...



(Graphics: Detlev Stalling, ZIB Berlin)

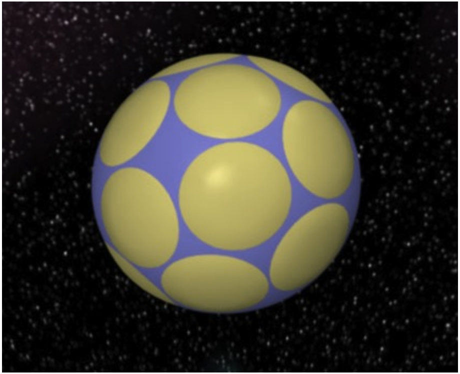
Packing by spherical caps

If unit spheres kiss the unit sphere S, then the set of kissing points is the arrangement on S such that the angular distance between any two points is at least 60°. Thus, the kissing number is the maximal number of nonoverlapping spherical caps of radius 30° on S.



Tammes' problem

Tammes' problem. How must N congruent non-overlapping spherical caps be packed on the surface of a unit sphere so that the angular diameter of spherical caps will be as great as possible



The Tammes problem

Let X be a finite subset of $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$. Denote

$$\psi(X) := \min_{x,y \in X} \{\mathrm{dist}(x,y)\}, \text{ where } x \neq y.$$

Then X is a spherical $\psi(X)$-code.

Denote by d_N the *largest angular separation* $\psi(X)$ with $|X| = N$ that can be attained in $\mathbb{S}^2$, i.e.

$$d_N := \max_{X \subset \mathbb{S}^2} \{\psi(X)\}, \text{ where } |X| = N.$$

L. Fejes Tóth (1943): $N = 3, 4, 6, 12, \infty$

K. Schütte, and B. L. van der Waerden (1951): $N = 5, 7, 8, 9$

L. Danzer (1963): $N = 10, 11$

R. M. Robinson (1961): $N = 24$

M. & Tarasov: $N = 13$ and $N = 14$

N	d_N
4	109.4712206
5	90.0000000
6	90.0000000
7	77.8695421
8	74.8584922
9	70.5287794
10	66.1468220
11	63.4349488
12	63.4349488
13	57.1367031
14	55.6705700
....	....
15	53.6578501
16	52.2443957
17	51.0903285

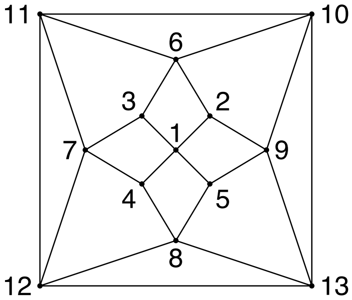
Contact graphs

The *contact graph* CG(X) is the graph with vertices in X and edges (x, y), $x, y \in X$ such that

$$\mathrm{dist}(x, y) = \psi(X)$$

N=13

The contact graph $\Gamma_{13} := \mathrm{CG}(P_{13})$ with $\psi(P_{13}) \approx 57.1367^\circ$



Tammes' problem for $N = 13$

The value $d = \psi(P_{13})$ can be found analytically.

$$2\tan\left(\frac{3\pi}{8} - \frac{a}{4}\right) = \frac{1 - 2\cos a}{\cos^2 a}$$

$$d = \cos^{-1}\left(\frac{\cos a}{1 - \cos a}\right).$$

Thus, we have $a \approx 69.4051^\circ$ and $d \approx 57.1367^\circ$.

Tammes' problem for $N = 13$

Theorem (M. & A. Tarasov). The arrangement of 13 points P_{13} in $\mathbb{S}^2$ is the best possible, the maximal arrangement is unique up to isometry, and $d_{13} = \psi(P_{13})$.

Theorem (M. & A. Tarasov) The arrangement of 14 points P_{14} in $\mathbb{S}^2$ is the best possible and the maximal arrangement is unique up to isometry.

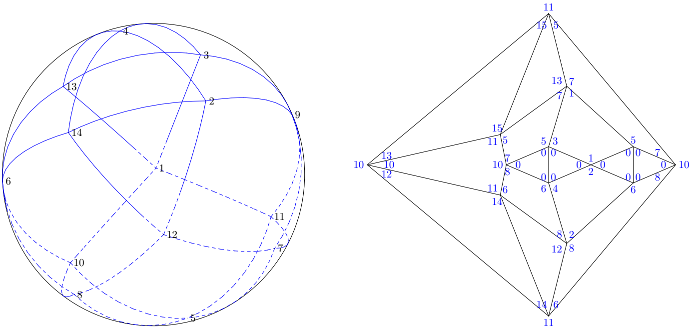


Figure: An arrangement of 14 points P_{14} and its contact graph Γ_{14} with $\psi(P_{14}) \approx 55.67057^\circ$.

Properties of the maximal contact graph G_N, $N = 13, 14$.

1. It is a planar graph with N vertices.
2. The degree of a vertex is 0,3,4, or 5.
3. All faces are polygons with m=3,4,5, or 6 vertices.
4. If there is an isolated vertex, then it lies in a hexagonal face.
5. No more than one vertex can lie in a hexagonal face.

Proof of Theorems

The proof consists of two parts:
(I) Create the list L_N of all graphs with N vertices that satisfy 1–5;
(II) Using linear approximations and linear programming remove from the list L_N all graphs that do not satisfy the geometric properties of G_N

The list L_{13}

To create L_{13} we use the program *plantri* (Gunnar Brinkmann and Brendan McKay). This program is the isomorph-free generator of planar graphs, including triangulations, quadrangulations, and convex polytopes. The program plantri generates 94,754,965 graphs in L_{13}. Namely, L_{13} contains 30,829,972 graphs with triangular and quadrilateral faces; 49,665,852 with at least one pentagonal face and with triangular and quadrilaterals; 13,489,261 with at least one hexagonal face which do not contain isolated vertices; 769,375 graphs with one isolated vertex, 505 with two isolated vertices, and no graphs with three or more isolated vertices.

Square packings

Obviously, the optimal packing in the torus could not be worse than the optimal packing in the unit square. Here are some results for the small number of disks in the square (Schaer & Meir (1965), Schaer (1965), and Melissen (1994)). Here d denotes the distance between the centers. Corresponding configurations are shown in Figure 1.

N	2	3	4	5	6	7	8	9
$d \approx$	0.586	0.509	0.500	0.414	0.375	0.349	0.341	0.333

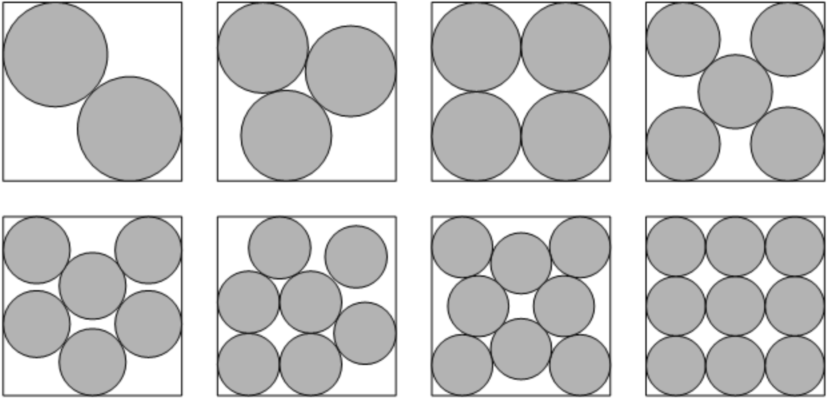


Figure: The optimal configurations for the square

The formal statement

The problem is: for a given number $N \geq 1$ of points, find the maximal $r \in \mathbb{R}^+$ such that N circles of radius r could be put on the square flat torus $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ without overlapping, or, equivalently, to find the maximal $d \in \mathbb{R}^+$ such that there are N points on the torus with pairwise distances not less than d (where $d = 2r$).

$N = 7,\, 8$

Theorem (M. & A. Nikitenko). *There are three, up to isometry or up to a move of a free disk, optimal arrangements of 7 points in* $\mathbb{T}^2$ *which are shown in Figures 7.1 - 7.3 where* $d(7) = \frac{1}{1+\sqrt{3}} \approx 0.3660$.

Corollary (M. & A. Nikitenko). *There is one unique, up to isometry, optimal arrangement of 8 points in* $\mathbb{T}^2$*, which is shown in Figure 8 where* $d(8) = d(7) = \frac{1}{1+\sqrt{3}} \approx 0.3660$.

Optimal packings of circles on a square flat torus: 7.1

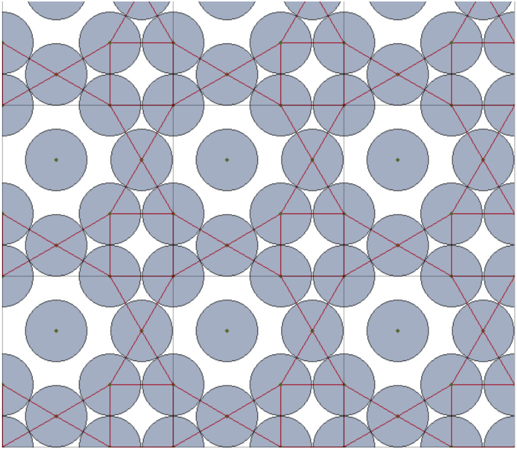


Figure: The first optimal configurations for N=7

Optimal packings of circles on a square flat torus: 7.2

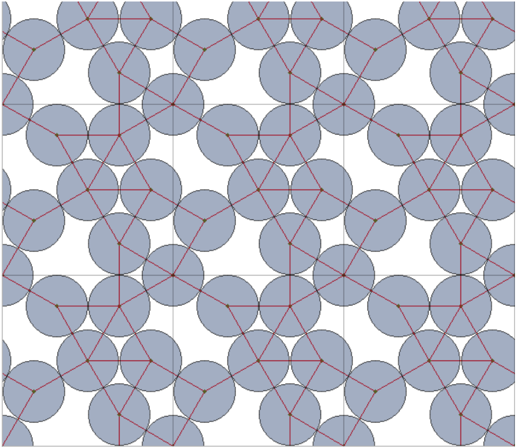


Figure: The second optimal configurations for N=7

Optimal packings of circles on a square flat torus: 7.3

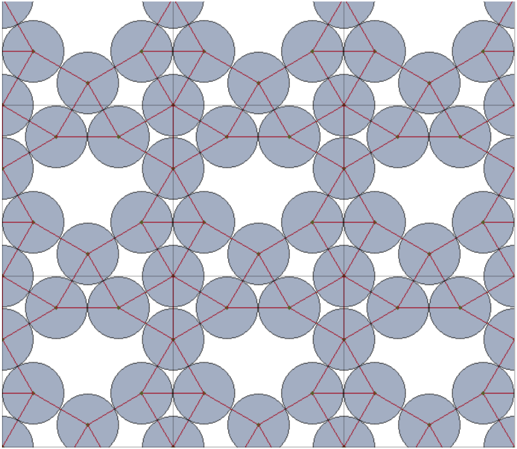


Figure: The third optimal configurations for N=7

Figure 8

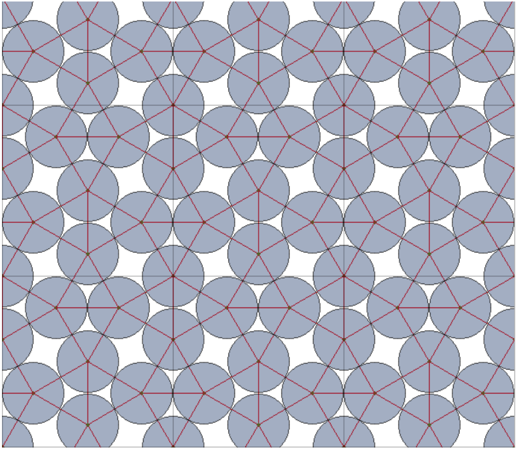


Figure: The optimal configuration for N=8

Fejes Tóth's bound

We say that X in $\mathbb{S}^{n-1}$ is a spherical φ-code if for any $x, y \in X$, $x \neq y$, we have $\mathrm{dist}(x, y) \geq \varphi$.

Denote by $A(n, \varphi)$ the maximum cardinality of a φ-code in $\mathbb{S}^{n-1}$.

In other words, $A(n, \varphi)$ is the maximum cardinality of a sphere of radius $\varphi/2$ packing in $\mathbb{S}^{n-1}$.

Fejes Tóth's bound

Theorem [L. Fejes Tóth, 1943]

$$A(3, \varphi) \leq \frac{2\pi}{\Delta(\varphi)} + 2,$$

where

$$\Delta(\varphi) = 3 \arccos\left(\frac{\cos\varphi}{1 + \cos\varphi}\right) - \pi,$$

i.e. $\Delta(\varphi)$ *is the area of a spherical regular triangle with side length* φ.

Fejes Tóth's bound

The Fejes Tóth bound is tight for $N = 3, 4, 6$ and 12. So for these N it gives a solution of the Tammes problem. This bound is also tight asymptotically.
However, for all other cases the Fejes Tóth upper bound is not tight. For instance, for $N = 13$ this bound is $60.92^\circ > 57.14^\circ$.

Fejes Tóth's bound

Theorem (Coxeter (1963) and Böröczky (1978))

$$A(n,\varphi) \le 2F_{n-1}(\alpha)/F_n(\alpha),$$

where

$$\sec 2\alpha = \sec\varphi + n - 2,$$

and the function F is defined recursively by

$$F_{n+1}(\alpha) = \frac{2}{\pi}\int\limits_{arcsec(n)/2}^{\alpha} F_{n-1}(\beta)\,d\theta,\ \sec 2\beta = \sec 2\theta - 2,$$

with the initial conditions $F_0(\alpha) = F_1(\alpha) = 1.$

Fejes Tóth's bound

Coxeter's bounds for kissing numbers $k(n) = A(n, \pi/3)$ with $n = 4, 5, 6, 7$, and 8 are
26, 48, 85, 146, and 244, respectively.

It also proves that

$$A(4, \pi/5) = 120.$$

Thank you