GROUPS OF ALGEBRAIC TRANSFORMATIONS OF AFFINE VARIETIES FINAL REPORT FOR THE THREE YEARS

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1. Results of this year

During this year, we have studied the interconnections between different results on normality in combinatorial algebraic geometry. Normality is a property of a finite set of points which means that the cone generated by these points has no "holes". Algebraically it means that the corresponding monomial algebra is integrally closed in its field of fractions, and for the algebraic geometry it means normality of the corresponding variety. There are many criteria for normality for algebras constructed from a given graph; for the set of circuits of a matroid; and many other. In our work in progress with I. Arzhantsev, we will show that the result for graphs of Simis-Vasconcelov-Villarreal and Ohsugi-Hibi, can be deduced from the criterion due to B. Sturmfels for toric ideals.

2. Papers

[1] with Fedor Bogomolov and Ilya Karzhemanov
Unirationality and existence of infinitely transitive models
Birational geometry, rational curves, and arithmetic, 2013, Ch. 4., p. 77–92. Boston: Birkhauser.

In this paper, we study unirational algebraic varieties and the fields of rational functions on them. We show that after adding a finite number of variables some of these fields admit a so-called infinitely transitive model. The latter is an algebraic variety with the given field of rational functions and an infinitely transitive regular action of a group of algebraic automorphisms generated by unipotent algebraic subgroups. We expect that this property holds for all unirational varieties and in fact is a peculiar one for this class of algebraic varieties among those varieties which are rationally connected.

3. Conferences, talks, schools

[1] Combinatorial Algebraic Geometry, Levico Terme, Italy, June 9-15, 2013, "Normality of maximal torus orbits closures in simple modules of algebraic groups"

[2] Second International Conference "Mathematics in Armenia: Advances and Perspectives", Tsaghkadzor, Armenia, August 25-31, 2013, "Normality of maximal torus orbits closures in simple modules of algebraic groups"

[3] Christmas meetings with Pierre Deligne, Independent University of Moscow, January 8-11, 2013, "Unirationality and existence of infinitely transitive birational models"

[4] 25th Summer Conference of International Mathematical Tournament of Towns, August 4-12, 2013, presentation of series of problems “Diophantine equations” (together with S.Grigoriev, A.Petukhov, and A.Semchenkov)

4. Work in international centers

- Centro Internazionale per la Ricerca Matematica, Trento, Italy

5. Teaching

[1] Lie Groups and Lie Algebras. Independent University of Moscow, II year students, February-May 2013, 3 hours per week, lectures and seminars.

Program

(1) Main definitions and examples: Lie group, Lie subgroup, Lie groups homomorphism, representations and actions of Lie groups.
(2) Orbits and stabilizers. Smooth structure on the set of cosets. Quotient groups.
(3) Left- and right-invariant tensors on a Lie group. Existence of the invariant volume form on a compact Lie group.
(4) Four ways to define the Lie algebra of a Lie group. Adjoint representation.
(5) Tangent homomorphism and tangent representation. Existence and uniqueness theorems for the homomorphisms of Lie groups.
(6) Exponential map. Description of all connected Lie groups with the given Lie algebra.
(7) Main classes of Lie groups and Lie algebras: solvable, nilpotent, simple, semisimple.
(8) Structure theory of semisimple Lie algebras: Cartan subalgebra, Cartan-Killing form, root system, Cartan matrix, Dynkin diagram.
(9) Classification of simple Lie algebras. Chevalley basis, Serre relations. Isomorphisms in small dimensions.
(10) Levi and Maltsev theorems.
(11) Introduction to the representation theory of Lie algebras: fundamental representations. Universal enveloping algebra. Representations of $\mathfrak{sl}(2)$. Highest weight representations. Weyl character formula.

[2] Probability Theory and Mathematical Statistics. Higher School of Economics, Management department, I year students, January-June 2013, 2 hours per week, seminars.

[3] Calculus III. Higher School of Economics & New Economic School, II year students, September-December 2013, 2 hours per week, seminars.

[4] Discrete Mathematics. Higher School of Economics, Philology department, I year students, September-December 2013, 4 hours per week, seminars.

[5] Algebra-3, Independent University of Moscow, II year students, September-December 2013, 2 hours per week, seminars.

6. Results of the whole period

In the project, we formulated the following problem (which was partially done by that moment). Let G be a simple algebraic group over an algebraically closed field, T any maximal torus in G and V any simple G-module. We are seeking for modules V such that for every $v \in V$, the closure $\overline{Tv}$ is a normal affine variety. In other words, we look for pairs (G, V), which are "good" in this sense. This question for adjoint modules was studied by J. Morand earlier, and then I continued this work for arbitrary simple modules of classical algebraic groups. During these three years, we (together with Ilya Bogdanov) have improved the methods, have simplified the proofs in the cases D_5 and D_6, and then it became possible to deal with the exceptional groups, especially E_6. Now we have a complete classification of all such pairs (G, V) and in all the other cases we present an orbit with non-normal closure.

Unfortunately, we did not find any new nice example of an affine infinitely transitive variety, since is was proved that (for an algebraically closed field) they are all unirational, and we started to study this question from the another point of view. However, this question is still interesting, especially over $\mathbb{R}$.

The third question mentioned in the project was infinite transitivity on each connected component for non-connected varieties over $\mathbb{R}$. In all the known results, if an algebraic variety is infinitely transitive, it can be achieved by the actions decomposing into actions of one-parameter unipotent subgroups. Since over $\mathbb{R}$ there exist irreducible non-connected varieties, we cannot have even one-transitivity of any connected group of automorphisms. So we use the refined definition: a variety is infinitely transitive on each connected component, if for any two $m_1+\ldots+m_s$-tuples of points, where $A_1, \ldots, A_{m_1}, B_1, \ldots, B_{m_1}$ are from the first connected component, $\ldots$, (equal number of points of both tuples on each connected component), there exist an automorphism mapping A_1 to B_1, etc., A_{m_1+1} to B_{m_1+1}, $\ldots$. We proved (together with F. Mangolte) the theorem which says that a suspension over an infinitely transitive on each connected component variety is also an infinitely transitive on each connected component variety. Note that the number of connected components of the given variety can increase on each step.