# REPORT ON THE DYNASTY FOUNDATION FELLOWSHIP 2012

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## Research

My research is focused on the PBW degeneration of various objects showing up in the Lie theory, both algebraic and geometric. The main objects here are irreducible representations of simple Lie algebras and (generalized) flag varieties. It turns out that the structures of the degenerate theory are important in representation theory, combinatorics, algebraic geometry and algebraic groups theory.

Roughly speaking, the idea of the PBW (Poincaré-Birkhoff-Witt) degeneration is to pass from a complicated nonabelian Lie algebra (Lie group) to a simpler abelian Lie algebra (abelian unipotent Lie group) of the same dimension. This procedure allows to endow the initial (classical) objects with new interesting structures, to construct new algebraic varieties with groups actions and to obtain geometric and representation theoretic realizations of classical combinatorial objects.

Let $\mathfrak{g}$ be a simple Lie algebra with a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ and the complimentary nilpotent subalgebra $\mathfrak{n}^-$, i.e. $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$. The PBW degeneration procedure produces a new degenerate algebra $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is the abelian Lie algebra with the underlying vector space $\mathfrak{n}^-$. Moreover, this algebra comes equipped with rich representation theory. Namely, to any irreducible highest weight $\mathfrak{g}$-module $V_\lambda$ one naturally attaches the corresponding degenerate object $V_\lambda^a$, which is a $\mathfrak{g}^a$-module. The space $V_\lambda^a$ has the following key features: it is of the same dimension and character as the classical counterpart; it has additional PBW-grading, which allows to define $q$-character of $V_\lambda$; it can be presented as a cyclic representation of a polynomial ring, i.e. it is isomorphic to a certain quotient of the polynomial ring modulo the ideal of relations. Together with G. Fourier and P. Littelmann we have described the structure of the representations $V_\lambda^a$ for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$. In particular, we have found the description of the defining ideal and a combinatorial formula for the $q$-characters of the degenerate representations. Also we have found new monomial bases for the modules $V_\lambda^a$ (and hence for the classical representations $V_\lambda$) in terms of the Vinberg polytopes. In our recent paper [8] we have proved that all the above mentioned results are still valid over the integer and hence over any field. Let us finish this part with a remark that the representation theory of the Lie algebra $\mathfrak{g}^a$ is reacher then that of $\mathfrak{g}$ and there are many open and important questions in this direction. The first steps in this direction are made in [5].

Let $G^a$ be the degenerate Lie group with the Lie algebra $\mathfrak{g}^a$. Hence $G^a$ is a semi-direct product of the Borel group $B$ with the normal abelian unipotent subgroup $\mathbb{G}_a^M$, where $\mathbb{G}_a$ is the additive group of the field and $M = \dim \mathfrak{n}^-$ is the number of positive roots of $\mathfrak{g}$ ($\mathbb{G}_a^M$, being the degeneration of $N^-$, is the Lie group of $(\mathfrak{n}^a)^-$). For a fixed dominant weight $\lambda$, we define the degenerate flag variety $\mathcal{F}_\lambda^a$ as the closure of the $G^a$-orbit of the highest weight line in the projective space $\mathbb{P}(V_\lambda^a)$ [1]. An important observation is that the degenerate flag varieties are the so called $\mathbb{G}_a^M$-varieties, i.e. the abelian unipotent group $\mathbb{G}_a^M$ acts of $\mathcal{F}_\lambda^a$ with open dense orbit.

For $G = SL_n$ the degenerate flag varieties are proved to be flat degenerations of the classical counterparts [1]. The varieties $\mathcal{F}^a_\lambda$ are singular projective varieties, enjoying in type $A$ explicit and simple description in terms of sequences of subspaces of a given vector space.

It has been shown in [3] that the degenerate flag varieties in type A ($G = SL_n$) are isomorphic to certain quiver Grassmannians. More precisely, let us take the equioriented quiver $Q$ of type $A$ and a $Q$-module $M$, isomorphic to the direct sum $P \oplus I$ of a projective and an injective representation. Then each degenerate flag variety is isomorphic to the variety of $\mathbf{dim}P$-dimensional $Q$-submodules in $M$ for a suitable choice of $P$ and $I$. This observation is important because of the two following reasons. First, it allows to generalize the known results on the degenerate flag varieties to a wider class of algebraic varieties – quiver Grassmannians of the special form. In particular, we have proved in [3] that all the quiver Grassmannians $Gr_{\mathbf{dim}P}(P \oplus I)$ are normal locally complete intersections. We have also constructed a desingularization of arbitrary quiver Grassmannians for Dynkin quivers [7], generalizing the corresponding construction for the degenerate flag varieties. Second, one can apply the tools of the theory of quiver representations to study the geometry of the degenerate flag varieties. In particular, it has been shown that there exists an algebraic group (a codimension one subgroup of the automorphism group of $P \oplus I$), which acts on $\mathcal{F}^a_\lambda$ with finitely many orbits, each orbit being an affine cell. Also this approach allows to describe the singular loci of the degenerate flag varieties [6].

Finally, let us mention the combinatorial structures showing up in the theory. It is well known that the Euler characteristics of the (complete) flag varieties for $SL_n$ are equal to $n!$. Moreover, one can easily write down the corresponding Poincaré polynomials. It turns out that the Euler characteristics of the (complete) degenerate flag varieties in type $A$ are given by the normalized median Genocchi numbers. These are classical combinatorial objects, showing up in various problems. Using our geometric approach we were able to derive several new combinatorial descriptions of these numbers and their $q$-analogues, defined as the Poincaré polynomials of the degenerate flag varieties. Moreover, together with G. Cerulli Irelli and M. Reineke we were able to derive explicit formulas for these numbers [3] and for their generating function [4]. Yet another combinatorial application comes through the study of the singular loci of the degenerate flag varieties. We have proved in [6] that the Euler characteristics of smooth loci of the degenerate flag varieties are given by the large Schröder numbers and the corresponding Poincaré polynomials are obtained via a natural statistics on the set of Schröder paths, counting the number of diagonal steps in a path.

## Papers

[1] $\mathbb{G}_a^M$ degeneration of flag varieties
Selecta Mathematica, 2012, Vol. 18 (3), pp. 513–537.

Let $\mathcal{F}_\lambda$ be a generalized flag variety of a simple Lie group $G$ embedded into the projectivization of an irreducible $G$-module $V_\lambda$. We define a flat degeneration $\mathcal{F}^a_\lambda$, which is a $\mathbb{G}_a^M$ variety. Moreover, there exists a larger group $G^a$ acting on $\mathcal{F}^a_\lambda$, which is a degeneration of the group $G$. The group $G^a$ contains $\mathbb{G}_a^M$ as a normal subgroup. If $G$ is of type $A$, then the degenerate flag varieties can be embedded into the product of Grassmanians and thus to the product of projective spaces. The defining ideal of $\mathcal{F}^a_\lambda$ is generated by the set of degenerate Plücker relations. We prove that the coordinate ring of $\mathcal{F}^a_\lambda$ is isomorphic to a direct sum of dual PBW-graded $\mathfrak{g}$-modules. We also prove that there exist bases in multi-homogeneous components of the coordinate rings, parametrized by the semistandard PBW-tableux, which are analogues of semistandard tableux.

[2] Systems of correlation functions, coinvariants and the Verlinde algebra
Funkts. Anal. Prilozh., 2012, Vol. 46 (1), pp. 4964

We study the Gaberdiel-Goddard spaces of systems of correlation functions attached to an affine Kac-Moody Lie algebra $\widehat{\mathfrak{g}}$. We prove that these spaces are isomorphic to the spaces of coinvariants with respect to certain subalgebras of $\widehat{\mathfrak{g}}$. This allows to describe the Gaberdiel-Goddard spaces as direct sums of tensor products of irreducible $\mathfrak{g}$-modules with multiplicities given by fusion coefficients. We thus reprove and generalize Frenkel-Zhu's theorem.

[3] With G. Cerulli Irelli and M. Reineke
Quiver Grassmannians and degenerate flag varieties
Algebra & Number Theory, 2012, Vol. 6 (1) (2012), pp. 165–194

Quiver Grassmannians are varieties parametrizing subrepresentations of a quiver representation. It is observed that certain quiver Grassmannians for type A quivers are isomorphic to the degenerate flag varieties investigated earlier by the second named author. This leads to the consideration of a class of Grassmannians of subrepresentations of the direct sum of a projective and an injective representation of a Dynkin quiver. It is proven that these are (typically singular) irreducible normal local complete intersection varieties, which admit a group action with finitely many orbits, and a cellular decomposition. For type A quivers explicit formulas for the Euler characteristic (the median Genocchi numbers) and the Poincare polynomials are derived.

[4] The median Genocchi numbers, Q-analogues and continued fractions
European Journal of Combinatorics, 2012, Vol. 33, pp. 1913–1918.

The goal of this paper is twofold. First, we review the recently developed geometric approach to the combinatorics of the median Genocchi numbers. The Genocchi numbers appear in this context as Euler characteristics of the degenerate flag varieties. Second, we prove that the generating function of the Poincaré polynomials of the degenerate flag varieties can be written as a simple continued fraction. As an application we prove that the Poincaré polynomials coincide with the $q$-version of the normalized median Genocchi numbers introduced by Han and Zeng.

[5] Degenerate $SL_n$: representations and flag varieties
arXiv:1202.5848, *to appear in Functional Analysis and Its Applications.*

The degenerate Lie group is a semidirect product of the Borel subgroup with the normal abelian unipotent subgroup. We introduce a class of the highest weight representations of the degenerate group of type A, generalizing the PBW-graded representations of the classical group. Following the classical construction of the flag varieties, we consider the closures of the orbits of the abelian unipotent subgroup in the projectivizations of the representations. We show that the degenerate flag varieties $\mathcal{F}_n^a$ and their desingularizations $R_n$ can be obtained via this construction. We prove that the coordinate ring of $R_n$ is isomorphic to the direct sum of duals of the highest weight representations of the degenerate group. In the end, we state several conjectures on the structure of the highest weight representations.

[6] With G. Cerulli Irelli and M. Reineke
Degenerate flag varieties: moment graphs and Schröder numbers
arXiv:1206.4178, *to appear in Journal of Algebraic Combinatorics.*

We study geometric and combinatorial properties of the degenerate flag varieties of type A. These varieties are acted upon by the automorphism group of a certain representation of a type A quiver, containing a maximal torus T. Using the group action, we describe the moment graphs, encoding the zero- and one-dimensional T-orbits. We also study the smooth and singular loci of the degenerate flag varieties. We show that the Euler characteristic of the smooth locus is equal to the large Schröder number and the Poincaré polynomial is given by a natural statistics counting the number of diagonal steps in a Schröder path. As an application we obtain a new combinatorial description of the large and small Schröder numbers and their q-analogues.

[7] With G. Cerulli Irelli and M. Reineke
Desingularization of quiver Grassmannians for Dynkin quivers
arXiv:1209.3960, *submitted to Advances in Matrhematics.*

A desingularization of arbitrary quiver Grassmannians for representations of Dynkin quivers is constructed in terms of quiver Grassmannians for an algebra derived equivalent to the Auslander algebra of the quiver.

[8] With G. Fourier, P. Littelmann
PBW–filtration over $\mathbb{Z}$ and compatible bases for $V_{\mathbb{Z}}(\lambda)$ in type $\mathtt{A}_n$ and $\mathtt{C}_n$
arXiv:1204.1854, *submitted to the proceedings of the conference Symmetries, Integrable Systems and Representations, Lyon, France, December 2011.*

We study the PBW-filtration on the highest weight representations $V(\lambda)$ of the Lie algebras of type $\mathtt{A}_n$ and $\mathtt{C}_n$. This filtration is induced by the standard degree filtration on $\mathrm{U}(\mathfrak{n}^-)$. In previous papers, the authors studied the filtration and the associated graded algebras and modules over the complex numbers. The aim of this paper is to present a proof of the results which holds over the integers and hence makes the whole construction available over any field.

**Scientific conferences and seminar talks**

[1] Conference "Classical and Quantum Integrable Systems", Russia, Dubna, January, 23 – January, 27
Talk "Abelianized representations of simple Lie algebras"
[2] Conference "Algebra and Geometry", Russia, Moscow, May, 4 – May, 9
Talk "PBW degeneration of flag varieties in type A"
[3] Conference "Symmetric Spaces and their Generalisations II", Italy, Trento, June, 25 – June, 29
Talk "Degenerate flag varieties"
[4] Conference "Lie Theory and quantum analogues", France, Marseille, April, 23 – April, 27

Talk “Degenerate flag varieties”

[5] Conference “Enveloping algebras and geometric representation theory”, Germany, Oberwolfach, March, 4 – Marrch, 10

Talk “PBW degeneration: representations and flag varieties”

**Teaching**

[1] Algebra. Independent University of Moscow, I year students, September-December 2012, 2 hours per week.

Program:

1. Groups, rings, fields: definitions, examples, properties.
2. Vector spaces and linear maps: bases, matrices, quotient spaces, linear equations, determinants.
3. Polynomials: divisibility, irreducibility, roots, the fundamental theorem of algebra, symmetric polynomials.
4. Group theory: homomorphisms, quotient groups, abelian groups, representations.
5. Linear maps: eigenvectors, Jordan normal form, exponent.
6. Dual spaces, bilinear and quadratic forms, euclidean spaces.
7. Tensor algebra: tensor product, exterior and symmetric algebras, determinant, Grassmann algebra.